Proof in the Elementary Grades

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The mathematics classroom is the preeminent context in which students can learn to rely on their own powers of reasoning. Research shows that, given a classroom environment in which they are encouraged to share their mathematical ideas, even students in the primary grades can and do engage in impressive mathematical reasoning (Ball & Bass, 2003; Lampert, 1990; Cobb at al., 1992). Investigators have found that young children are able to explore and discuss the regularities they observe in the number system, articulate the generalizations they see, and deal successfully with such questions as, "How can we know this is true for all cases?" Thus, at an early age, these students are already engaged in the process of proving claims of generality.

This paper investigates the possibility and actuality of proof in the elementary grades. How do young children consider mathematical claims that apply to a general class? Which criteria for proof are appropriate to elementary students, yet would support distinctions essential in the later grades? And how does such work on proof support the work that is already at the heart of the elementary mathematics program?

The Context of Our Work

The ideas presented in this paper come out of the work of both the

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"algebra team" responsible for revising a component of the K-5 curriculum, *Investigations in Number, Data, and Space* and the creators of the *Developing Mathematical Ideas* professional development materials. Since 1993, we have worked in collaboration with groups of teachers to investigate students' mathematical thinking in classroom contexts (Schifter et al., 1999). More recently, we have focused on how early algebra can arise quite naturally out of the work of the elementary classroom and how teachers can encourage and develop it further (Bastable & Schifter, in press; Schifter, 1999, Schifter et al., in press).

Monthly meetings of project staff with collaborating teachers were organized around a set of mathematical tasks designed to help teachers explore arithmetic generalizations that might arise in elementary classrooms. In addition, teachers read and discussed cases—reflective and usually detailed descriptions of classroom episodes—produced in earlier projects.

As teachers engaged together on the mathematics and on the cases, they refined their understanding of mathematical generalization, analyzed children's thinking, and considered which classroom activities would best support children across the grades as they take up these ideas. The teachers introduced these or related mathematics activities to their students, then wrote their own cases—meant to be shared with the group—documenting the resulting classroom process. They reported on: (1) their students' thinking as this was reflected in classroom conversation, (2) the ways in which representations were used by their students, and (3) the questions this episode brought up for them about their own teaching practice. Staff members read and responded to each case, highlighting in particular the mathematical-conceptual issues in play.

These cases, together with videotapes from the classrooms of a small subset of teachers in the group, provided data we used to refine our own thinking about the key algebraic ideas to be addressed in the curriculum, the development of those ideas across the grades, and the classroom tasks that could draw them out. The data also quite quickly led us to the question, what counts as justification of a mathematical generalization in the elementary grades?

The work of generalizing and justifying emerges quite naturally from ordinary classroom activities in which children engage in their study of arithmetic. Once students have noticed and described a regularity in the number system, the questions *why does this pattern hold?* or *will this always work?* lead to the search for justification. As we study the nature of arguments presented by K-5 children, we have asked: What forms of explanation and justification do children offer as they begin to realize the problematic nature of making claims about infinitely many numbers? Through what means do they address the problem of making such arguments? What aspects of students' arguments align with ideas about proof they will encounter in later grades?

Students' Responses to the Challenge of *All*

Consider the following set of exchanges, a composite drawn from several third-grade classrooms:

> Students have been studying odd and even numbers and have established that one way to ascertain that a number is even is if that number of objects can be arranged in pairs with nothing left over. A number is odd if that number of objects, when arranged in pairs, has one left over without a partner. Today, in the course of a discussion, the class conjectured that the sum of two even

numbers is even. When the teacher asked students to explain how they knew this will *always* be the case, no matter what two even numbers are added, they offered the following responses.

Paul: I know the sum is even because my older sister told me it always happens that way.

Zoe: I know it will add to an even number because 4 + 4 = 8 and 8 + 8 = 16.

Juan: Also, 6 + 12 = 18 and 32 + 20 = 52.

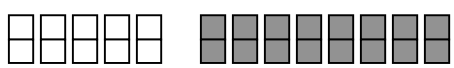
Eva: We really can't know! Because we might not know about an even number and if we add it with 2 it might equal an odd number!

James: We can never know for sure because the numbers don't stop.

Claudia: We don't know because numbers don't end. One million plus one hundred. You can always add another hundred.

Melody: Your answer will be even because you are using even numbers.

As Melody spoke, she pointed to arrangements of cubes in front of her.



Then she continued.

Melody: This number is in pairs (pointing to the light colored cubes), and this number is in pairs (pointing to the dark colored cubes), and when you put them together, it's still in pairs.

Ms. Emerson has challenged her students to address a problem central to the doing of mathematics: How can you defend a claim that applies to an infinite number of instances? Her students' responses represent four categories that might be expected of a group of third graders.

- Appeal to authority
- Inference from instances
- Claims about an infinite class cannot be proven
- Reason from representation or story context

The first category is illustrated by Paul who is ready to accept the claim by appeal to an authority, in his case, authority ceded to his older sister. Students might accept a claim because a teacher told them, because it's "in the book," or because the "smart" kids in the class say it's true. As one moves through school and is socialized into our culture, there are many things, including mathematical definitions and conventions, that students must accept as given. However, the mathematics classroom provides a context for students to learn to rely on their own powers of reasoning.

The second category of response is reliance on particular instances to support a general claim. Zoe attempts to demonstrate the claim that the sum of two even numbers is even by adding 4 + 4 and 8 + 8. Students are often quite satisfied with this means of justification: "It works in the cases I tried, so it must be true." In fact, research reports indicate that even many college students are satisfied to accept a general claim on the evidence of a few examples (Harel & Sowder, 1998, in press; Knuth, et al., 2002; Martin & Harel, 1989).

Juan may have recognized that, in choosing equal addends (doubles), Zoe has picked a special class of example, and the claim might be true only for that class. (In fact, by an alternative definition of even, the double of any whole number is even.) So Juan uses unequal even addends to test the claim.

In fact, Juan's strategy illustrates a useful habit when testing a general claim. By choosing specific instances of different types, one might succeed in refuting a claim. Finding examples in support of the claim points in the direction of its truth and may yield greater insight into the claim, itself. But ultimately an accumulation of instances is not adequate mathematical justification.

A third category of response is found among those children, illustrated here by Eva, James, and Claudia, who recognize the inadequacy of relying on particular instances to justify a general claim and thus conclude that certainty is impossible. As students become aware that numbers are infinite, they also come to see that a conjecture cannot be proved by testing examples, since one counterexample is enough to refute the claim. And since "numbers go on forever," it's impossible to check every instance. Eva points out that there *might* be one even number out there, which they don't know about, but when 2 is added, yields an odd number. She reasons that, since they can't test all numbers, they'll never know.

Melody takes a different approach, representing the fourth category of response: using a representation to make an argument based on reasoning. Melody refers back to the definition of even number and shows that one number represented by pairs of cubes, joined with another number of paired cubes, yields a larger set of pairs.

However, although Melody doesn't speak about particular numbers, her representation is necessarily of specific numbers of cubes. How do we know that she is, in fact, making an argument about the sum of any two even numbers, rather than the particular numbers 10 and 16? She offers further evidence when the teacher questions her, and Melody again explains her reasoning.

Melody: Because 2s don't get to odds. If they're two even numbers, they're both counted by 2s, and if you put them together, you keep counting by 2s and that always equals an even number.

Eva: So she's saying she already knows it that it always equals even.

Thus, even though Melody points to particular numbers of cubes, she says she is thinking about *any* pair of even numbers. The same argument would work, even if she had an arrangement of 6 cubes and 4 cubes, or 8 cubes and 22 cubes. That is, the conclusion follows from the structure of her representation, rather than the specifics of the instance she happens to have chosen.

What Constitutes Proof at the Elementary Level?

Consider a slight variation of Melody's argument as compared to one that might be offered by a mathematician.

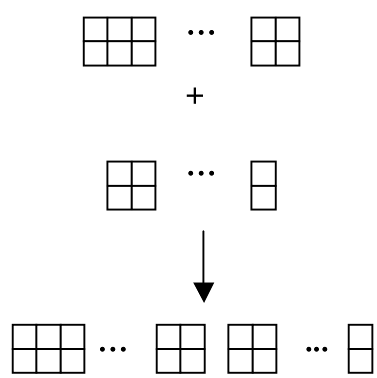
<u>Proposition</u>. If m and n are even numbers, then their sum, $m+n$, is an even number.	Two even numbers added together give an even number.
<u>Proof</u>. If m and n are even, then, by definition, there are some integer k and some integer j such that $m = 2k$ and $n = 2j$. Then $m+n = 2k + 2j$. Applying the distributive law, $m+n = 2(k+j)$. Thus, $m+n$ is equal to 2 times an integer, establishing that the sum is an even number.	Since they're two even numbers, they're both counted by 2s.  When you put them together, you keep counting by 2s.

Figure 1 –Mathematician's proof A variation of Melody's proof

Each argument states a premise (two even numbers are added) and argues to a conclusion (the sum is even). Both show how the conclusion follows from the premise, each step justified by a definition, fact, or principle already established. The mathematician relies on the laws of arithmetic, definitions that apply to the domain of integers, and algebraic notation to communicate the argument. Whereas Melody employs a spatial representation that embodies a definition of even number applicable to the domain of counting numbers and demonstrates the action of addition as the joining of two sets to show how the conclusion follows from the premise.

The variation of Melody's proof in Figure 1 uses dots to indicate any number of pairs of cubes, representing *any* two even numbers. However, the dots were inserted by the author of this paper to communicate how the spatial representation can accommodate the infinite class even numbers. This device increases the complexity of the argument without necessarily increasing its persuasiveness to an elementary classroom (Monk, in preparation).

Clearly, the tools of the mathematician's proof are not generally available to children in the elementary grades. At this level, most students are still coming to understand the kinds of actions that are modeled by the four basic operations. The laws of arithmetic cannot be the basis of their justifications when these laws are still in question for them. Indeed, what mathematicians call the commutative law, for example, might arise in a classroom as a conjecture to be proved. Nor is algebraic notation typically available as means for expressing generality.

However, young children *are* capable of proving claims of generality. Reasoning from visual representations to justify general claims, as Melody does, appears to be accessible, powerful, and generative for students. Our work has convinced us that argument from representation is the most effective route to establishing general claims in the elementary classroom, and we refer to such arguments as "representation-based proofs." The next question then: what criteria for representations can be offered for such arguments? Through studying examples of student proofs, we have identified three:

1. The meaning of the operation(s) involved is represented in diagrams, manipulatives, or story contexts.
2. The representation can accommodate a class of instances (for example, all whole numbers).
3. The conclusion of the claim follows from the structure of the representation.

We hypothesize that if this form of reasoning were to be explicitly encouraged in the elementary grades, it would enhance students' work with proof in later grades. Specifically, students would have experience with the ideas that 1) relying on specific instances to prove a general claim is logically flawed and 2) proving the truth of a claim that applies to an infinite class is nonetheless possible.

Studying Computation and Representation-Based Proof

One common objection to working on the ideas of proof in the elementary grades is that the curriculum is already very full. It is unreasonable to assume that more and more content should be moved into the elementary grades because students do so poorly with that content in the older grades. Our research demonstrates that working on representation-based proof actually can enhance the study of the content everyone agrees is at the heart of the elementary curriculum.

Two vignettes have been selected to illustrate this point. These classroom scenes come from a time when teachers were experimenting, trying to figure out what it means to work on proof with their elementary students.

The first vignette is taken from Margie Riddle's fourth-grade classroom. Here, when a computational question arose among her students, Riddle provided time for her students to explore the issues that underlay it. In that context, they devised a proof to resolve for themselves a mathematical relationship they initially found surprising.

The second vignette comes from Karen Schweitzer's first- and second-grade combination class. In this case, Schweitzer asked her students to come up with a proof for a proposition that everyone in the class already accepted. And once the proof was before them, they could extend the generalization and apply it to other computational contexts.

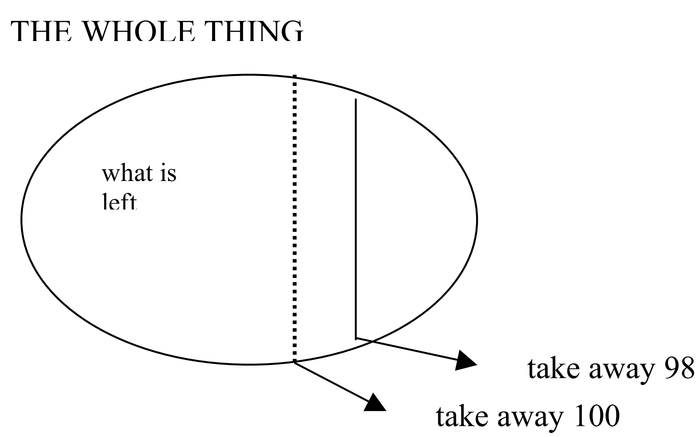
Proof as an aid to understanding: An example from a grade 4 classroom

As part of her morning routine, Margie Riddle had given her fourth-grade class some subtraction problems. Since her students were in the middle of a science project in which they weighed apples as they gradually dried out, she set the problems in that context. Included among them were 145-100 and 145-98. As the children began to consider the latter problem, Riddle realized here was an opening that held much potential for learning, and so she deferred further discussion until the math lesson later in the day.

What Riddle had seen was this: Many of the children realized there was a connection between the two problems. However, after calculating 145-100=45 and before actually solving 145-98, they weren't sure if the answer to the second problem would turn out to be 2 more or 2 less than 45. Once they did solve it and they knew the answer was 47, they wondered, why was it 2 *more*? After all, when they changed 100 to 98, they *subtracted* 2, so why *add* 2 to get the right result? Their puzzlement, Riddle felt, could lead her students to a deeper appreciation of the meaning of the operation of subtraction.

When the class returned to the problem later that day, several children offered their ways of finding the answer to the problem: 145-98=47. Riddle describes what happened next.

> Brian was waving his hand in the air, insisting on explaining his thinking, too. He struggled to find the words. "It goes with the problem before," he declared. "It's like you've got this big thing to take away and then you have a littler thing to take away so you have more. Can I draw a picture?"
>
> I nodded, and he came up to the blackboard, thought for awhile, and then drew a big blob like this:



> "See, this is the apple at first," he explained. "And you take some away and have some left. Then you take away 98 grams instead, so it's

over here." It appeared to me that Brian had a very clear mental image that was helping him think his way through the problem, but that he was having a hard time communicating it to us.

However, his classmates were watching and listening fairly intently, and suddenly, inspired by his presentation, Rebecca said excitedly, "Yeah, it's like you have this big hunk of bread and you can take a tiny bite or a bigger bite. If you take away smaller you end up with bigger."

"Do you think this will always be true?" I asked. "I think so," she answered.

During the discussion up to this point Max had been quiet, although he had correctly figured out early on that the answer was 47. However, he seemed inspired by Rebecca's explanation and Brian's picture. He continued further with the thinking that was unfolding when he raised his hand and said, "Yeah, the less you subtract, the more you end up with. AND …" he continued with great emphasis, "in fact the thing you end up with is exactly as much larger as the amount less that you subtracted."

These fourth graders' exploration of a generalization began with a simple arithmetic problem: 145-98. However, since that problem was juxtaposed with a similar problem, 145-100, students were immediately drawn to compare them. And although their initial question was formulated in terms of specific numbers—since 145-100=45, and since 98 is 2 less than 100, shouldn't 145-98 be 2 less than 45?—a more general question lay just underneath—if you decrease a number in a subtraction problem by a certain amount, shouldn't your answer decrease by the same amount?

The class discussion about this question began with students' strategies for calculating 145-98. For example, Lorenzo took the class through the steps of adding on from 98 to 145 to show that the missing part must be 47.

> "I did 40+98=138. I know that because
>
> 98 + 10 = 108
>
> 108 + 10 = 118
>
> 118 + 10 = 128
>
> 128 + 10 = 138. Then
>
> 138 + 7 = 145. So the answer is 40 and 7. It's 47."

Although Lorenzo's (and other students') explanation verified that the answer to 145-98 is 47, not 43, it did not satisfy the class's desire to understand *why* the numbers worked out that way.

But Brian's "blob" took the class into different territory. His representation illustrating the action of subtraction—removing a part from a whole—showed not only *that* the result of 145-98 (the amount to the left of the solid line) is greater than the result of 145-100 (the amount to the left of the dotted line), but *why* it has to work out that way.

Although Brian's representation was labeled with specific numbers (98 and 100), his classmates interpreted it in general terms. Indeed, if we ignore the labels "98" and "100," Brian's representation satisfies our three criteria for proof: 1) Subtraction is represented by removal of a part from a whole. 2) The whole can be any positive value, the subtracted amounts any positive values less than the whole. 3) The premise that two different amounts are subtracted from the same whole is shown, and the representation demonstrates the conclusion that must follow—when the lesser amount is subtracted, the amount remaining is larger.

Rebecca read the representation in these general terms and declared, “If you take away smaller, you end up with bigger.” Max offered a more precise statement of the generalization: “The less you subtract, the more you end up with, and in fact the thing you end up with is exactly as much larger as the amount less that you subtracted.”

However, Max’s elegant statement was not the end of the lesson. After all, there were other children in the classroom with thoughts and questions of their own. Riddle writes:

> I asked if anyone else wanted to comment, and Riley raised his hand. Having experienced several moves in the past two years of school, he is a student who is caught between traditional procedures that he has trouble remembering and doesn't understand, and trying to catch up to his classmates who are more accustomed to figuring out strategies for themselves. He often feels lost during math discussions, but today he seemed eager to get involved. “I used to think it was 43," he said, "but now since I saw it on the calculator and heard everyone talk, it's 47, but I don't get why."
>
> Often Riley was completely lost during these discussions, but this time I suspected he was actually at the brink of understanding. "Let's think about a different context," I suggested. "Pretend ... you have 145 pennies, and, the first time I take 100 from you. Now go back to you have 145 pennies again, but this time I take 98."

> A huge smile broke across RiIey's face. "Oh, now I see," he said happily, confirming my expectations. He added, "It's like you replay it in your mind, and now it makes sense." Like Brian, he now seemed to have a mental image that helped him solve the problem.

In this classroom episode, the impulse to prove grew naturally from the mathematical question these fourth graders posed to themselves. This episode illustrates that the work of proof need not be an add-on to an already full curriculum. Rather, it can support students' understanding of calculation and how the number system behaves under the four basic operations.

Furthermore, Riddle's example shows that the work of proof should not be reserved for those students who tend to excel in mathematics. In this case, the proof developed from whole-class discussion, engaging students who represented the entire spectrum, from those who excel in mathematics to those who often struggle. Brian (who, Riddle later explained, more often was quiet during mathematics discussions) provided an image, but struggled with the language to articulate the idea it represented. Rebecca offered a straightforward verbalization of the generalization along with another image from students' everyday lives. Max articulated the generalization more precisely.

Riley, a struggling student, couldn't quite follow the discussion as it played out, but did develop a sense that he could understand. After he asked for help, Riddle talked him through the ideas, now with a different context, which satisfied Riley. Although he may not have followed the ideas through to the generalization, he could make sense of why 145-98 must result in an answer 2 more than 145-100.

Proof as a route to conviction: An example from a grade 1-2 classroom

In solving particular subtraction problems, Margie Riddle's fourth-grade students had come across a question that confused them all—if you *decrease* by a certain amount a number you're subtracting, do you *add* or *subtract* that amount to/from the difference? Their proof not only showed them *that* you add that amount, but it also explained *why*.

But a proof can serve other functions, as well. For example, a proof might be used to convince someone else of a claim, even if you are already convinced of its truth yourself.

First- and second-grade teacher, Karen Schweitzer, had a series of discussions with her students about what happens to the sum when the order of two addends changes (what they will later learn to call the commutative property of addition), and by now they were no longer counting to test it out; they were all convinced that the sum remains the same. But beyond that, when she asked her students to explain their thinking or to say why they felt so sure, they didn't have much to say. Schweitzer had a hunch there was more to be mined from this question. And so one day she returned to the idea that the order of the two addends doesn't matter, the total stays the same, and told her students that the next day they would spend time showing how they knew this was true.

But what could motivate this exercise? Unlike Riddle's fourth graders who had formulated their own puzzlement, Schweitzer's students were already convinced of their generalization; they felt no need to explore it further. And so Schweitzer called upon another function of proof: to convince others of the truth of a claim.

Schweitzer explained to her students:

> [T]o prove it meant to convince someone. For example, if the

principal came in and they had to convince him that what they said about the order was true. I told them they could use any tools they needed to help them explain and I listed cubes, diagrams and number lines as possible options. With that, I sent them off in pairs to work on their proofs.

We came back together as a group after about 15-20 minutes of the children working in pairs. Some children brought with them cubes or base-10 blocks, and some brought written work. I wanted to make sure that they really had the idea of needing to convince someone. I thought this might help us get past the notion that we all know this so what else is there to say. I began the conversation with this reminder. "I want you to make sure that when you are doing your explaining, I want you to pretend that Mr. Valen is standing here and he doesn't believe you so you've got to be really careful to convince him. Don't assume anything. Say all the things you need to say and really convince him, like prove it."

I chose Kathleen to start us off because she is someone who often fits into the category of saying something is true but not being able to say why, but this time she had found a way to express why. She brought a clear diagram with her, in which she had drawn 10 cubes and 20 cubes, showing that they could be added in either order and the total remains the same.

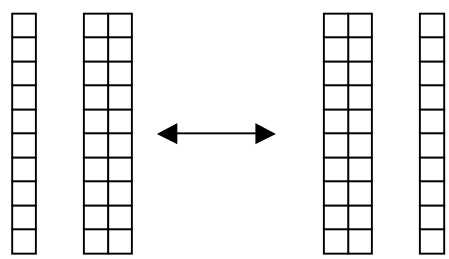
Kathleen: If you take a 10 and a 20 and then you switched them around, it will just equal the same thing. You put the 10 in one place and then you put the twenty, and you put the 20 where the 10 was.

Schweitzer posed several questions to Kathleen, asking her to point to her picture as she spoke, connecting the visual image to her words. And although the discussion had been particular to the image of 10 cubes and 20 cubes, soon Kathleen spoke in terms of a generalization.

> Kathleen: Because it's like any way you switch it, it would just be the same thing. Like, any number you could probably do it with. You could probably do it with any number.

Although Kathleen left room for doubt ("You could *probably* do it with any number"), she now extended her thinking beyond 10 and 20 to consider *any* number. Schweitzer emphasized this move by posing a question to the class.

> Teacher: So are you saying that you think it would work with any number? Like with any two numbers you could switch them and it won't change the answer?
>
> Andrew: Any, any number.



Once the cubes in Kathleen's image move from representing the specific numbers 10 and 20 to representing *any* two addends, it now satisfies the criteria for proof: 1) Addition is represented by the joining of two sets. 2) The representation can accommodate any two whole numbers. 3) The representation shows that switching the placement of the two quantities does not change the total.

When the question of the order of addends arises in other classrooms, students often produce an image similar to Kathleen's. As they exchange the position of the two sets, students explain, "You don't add any more and you don't take any away, so the total stays the same."

Now, with this image solidly in mind, the students in Schweitzer's class are ready to extend their generalization.

> Teacher: Andrew, you are agreeing with any, any number you can do that?
>
> Kirsten: Like 17 and 33.
>
> Nathan: And you don't just have to use two numbers.
>
> Teacher (I decided to focus on the any number part of the discussion before we moved on to more than two numbers): Like any two numbers, like 17 and 33.
>
> All of a sudden three or four children were talking at once. I tried to get all of the ideas heard.
>
> Teacher: So Kirsten said you're just switching to a different spot. Molly, what did you say?
>
> Molly: I said you didn't have to use only two numbers.
>
> Teacher: You didn't have to use only two numbers.
>
> Kirsten: I know you could use 10 and 5 and 3.
>
> Teacher: Wow, you could use three numbers. Like 10 and 5 and 3?
>
> Molly: You could use eight numbers.
>
> Kirsten: It doesn't matter.
>
> At this point there were children joining in and talking all at once saying that you could use more than two numbers and in fact that it didn't matter how many numbers you used. Even a billion numbers!
>
> The lesson had begun with a discussion of what happens when the order

of two addends is reversed. Although the students in this class had already been convinced that the total remains constant, their teacher had insisted that they develop a proof—an argument that would convince the school principal had he been standing at the door and been doubtful. Whereas the students created demonstrations that necessarily represented specific numbers—10 and 20 cubes, tally marks, circles, etc.—these could stand for *any* two numbers: when the quantities are switched around, the total is unchanged. With that image in mind, the students extend the generalization even further: you can start with any number of addends—"Even a billion numbers!"—and changing the order will not change the sum.

Although several members of the class had enthusiastically reached this conclusion, Schweitzer felt there was more to explore. By continuing the discussion with different representations offered by the students, more children could be brought into the argument, and those who were already convinced could consider applications of this idea. As the class discussion continued, students looked specifically at the sum of 4 + 6 + 2—an easy problem for students who were fluent with combinations that made 10 and how to add any one-digit number to 10. But now they understood that if they came across those same addends in a different order—4 + 2 + 6 or 6 + 2 + 4—they already knew the total without have to tackle a more challenging calculation.

Another group showed their representation of 140 with base-10 blocks—one flat of 100 and four sticks of 10.

> Corey: You're just switching them around [switching the position of the blocks on the carpet] and not putting any more on or taking any away. You're not adding some or you're not taking any away. You're just

switching them around and putting them in different spots.

Teacher: Okay.

Marissa: And you also can break the numbers up only if you don't take any or add any more on.

Teacher: Marissa, can you say more about that? You can break the numbers up as long as you don't . . .

Scott: Yeah.

Marissa: . . . take any away or add any more on.

Corey went on to show that the blocks could be arranged as 10 + 10 + 100 + 10 + 10, or as 10 + 10 + 10 + 10 + 100.

These students had already agreed that you can change the order of *any number* of addends of *any* size without changing the total, but now the base-10 representation moved them into decomposing multi-digit numbers and reordering the components. Joining the idea of reordering addends with place-value decomposition will be an essential skill in developing computational fluency with multi-digit numbers.

Conclusion

The episodes chosen for this paper illustrate what proof can look like in the elementary classroom. Elementary students take on generalizations about the number system that apply to an infinite class and, through the use of representations that embody the action of the relevant operations, devise arguments to prove the truth of the generalizations in question. The claims proven in these examples are:

- The sum of two even numbers is even.
- The less you subtract, the larger the difference.

- The order of addends does not affect the sum.

The cases written by teachers in our project show students proving a wider set of claims. For example:

- The sum of two odd numbers is even; the sum of an odd and an even is odd.
- The order of factors does not affect the product.
- In an addition problem, if you subtract a certain amount from one addend and add it to the other, the sum remains constant.
- In a subtraction problem, if you subtract (add) the same amount from (to) both numbers, the difference remains constant.
- If you double one factor and halve the other, the product remains constant.
- The factor of a number is also a factor of that number's multiples.
- In a multiplication problem, you can decompose one factor, multiply the parts by the other factor, and add the sub-products.

The goal of having students prove such claims is not merely one more piece of content for teachers to squeeze into an already full agenda. Rather, as the examples presented in this chapter illustrate, the challenge to prove generalizations about numbers and operations engages students in many facets of mathematical proficiency (Kilpatrick, et al.). Through these discussions, students are developing richer understandings of the meaning of the operations and their relationships, which in turn support greater flexibility with computational procedures.

Nor is the challenge to prove an enrichment activity reserved for one category of student. As one teacher wrote, "When I began to work on generalizations with my students, I noticed a shift in my less capable learners. Things seemed more accessible to them." When generalizations are made explicit—through language and through representations used to justify them—they become available to more students and for these students become the foundation for greater computational fluency. Furthermore, the habit of creating a representation when a mathematical question arises supports students in reasoning through their confusions. They come to see mathematics as sensible and develop confidence in their own efficacy.

At the same time, students who generally outperform their peers in mathematics find this content challenging and stimulating. The study of number and operations extends beyond efficient computation to the excitement of making and proving conjectures about mathematical relationships that apply to an infinite class of numbers. As one teacher explained, "Students develop a habit of mind of looking beyond the activity to search for something more, some broader mathematical context to fit the experience into."

Finally, at the same time that such an approach to proof deepens students' understanding of the number system, it engages them in a process central to mathematics. As they experience a variety of modes of coming to believe the truth of a claim, reasoning becomes the standard for its acceptance. They learn to privilege this kind of reasoning over appeals to authority or testing instances.

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