

Grades
8–9

Algebra: Non–Linear Relationships

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Finding and Describing Non–Linear Patterns

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Toothpick Squares

1

Overview

Mathematical Focus

- ▶ Patterns in visual representations
- ▶ Matches between arithmetical patterns and visual patterns
- ▶ Rules for finding any element in a pattern, using both words and expressions
- ▶ Input/output tables and graphs as representations of patterns
- ▶ Quadratic relationships in the form $y = x^2$ and $y = a(x^2 + x)$

In this activity, students begin by constructing a series of squares with toothpicks or by drawing the squares on paper. They continue drawing or constructing larger and larger squares until they can predict the number of small squares within the next larger square, and the number of toothpicks needed for the next square in the sequence.

Students construct squares and record the numeric data in a table. They plot input/output pairs as points on a coordinate graph, and compare the two relationships. Students describe the patterns they discover in terms of mathematical rules.

Preparation and Materials

Before the session, gather the following materials:

- ▶ 100 toothpicks (or popsicle sticks, etc.)
- ▶ Graph paper, pencil, and straight edge

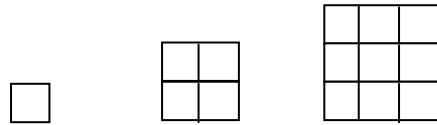
Notes

You may use toothpicks, popsicle sticks, or something similar for this activity. Whatever material you choose to use, all the pieces must be the same length.

Activity

Part 1: How Many Toothpick Squares?

- 1 Have students start by constructing the first three toothpick squares with toothpicks (popsicle sticks, etc.).



Teaching Tip

The pattern relating the number of small squares to the length of the large square, $N = L^2$, is the most basic form of a quadratic relationship. The most general type of quadratic relationship has the form $y = ax^2 + bx + c$. (In this simple case of $N=L^2$, $a = 1$, $b = c = 0$.) The shape of its graph (half of a parabola) is typical of the graphs of all quadratic functions.

Ask students if they know how to build the next larger square in this sequence. Ask them how many small squares will be inside the larger one.

After students have made their predictions, they should build the next two squares (#4 and #5), and check their predictions against the squares.

- 2 Ask students to draw the first five toothpick squares on graph paper. As students continue with the activity, it may become easier to have them draw squares on graph paper than to build them with toothpicks.

- 3** Remind students that when they studied Linear Functions, they used input/output tables as one way to represent and compare different functions. Have the students create input/output tables with the input being the length of each large square and the output being the number of small squares inside each large square.

Input: Length L , the number of toothpicks along one side of a square	Output: S , the number of small squares inside each larger square
1	1
2	4
3	9
4	16
5	25
6	
7	

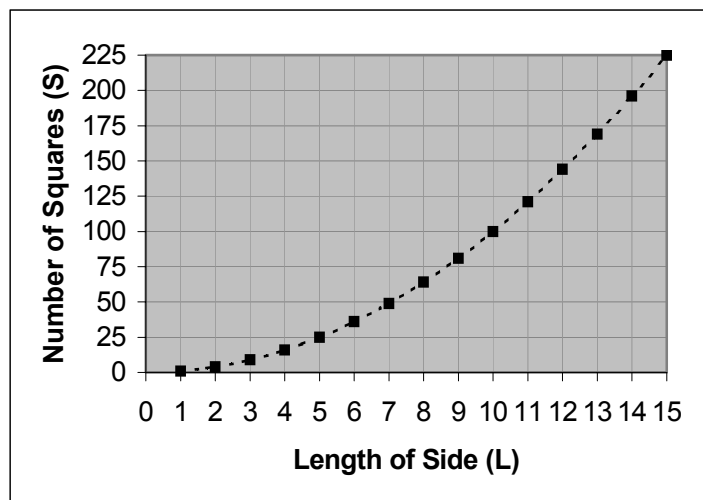
- 4** Encourage the students to look for patterns in the data. Ask: Can you predict the number of small squares inside a large square with 6 toothpicks on each side? With 7 toothpicks on each side?

If necessary, have students continue constructing squares until they notice and are able to describe the pattern. Students should notice that the number of small squares increases by an ever-larger amount. Ask them to predict how many more squares will be in the *next* large square. (The number of small squares in the *next* square is $2L + 1$ more than the present square, but students may not yet see this pattern.)

- 5** Ask students to describe a rule for predicting the number of small squares inside a large square with any length side.
- ♦ Ask students how many small squares will be in the tenth and the fifteenth large squares.

- ♦ Ask them to describe the length of the large square and the number of small squares in such a way that they feel they can find the number of small squares in any large square if they know its length. [Sample response: The number of small squares is equal to the length of one side multiplied by itself (or the length of one side square)].
- ♦ Ask students to write a mathematical expression describing this rule, using S to stand for the number of squares and L to stand for the length of the large square (for example, $S = L \times L$ or $S = L^2$).

6 Explain that the next step is to make a graph. Ask students to set up a piece of graph paper with L , the input, on the horizontal axis, and S , the output, on the vertical axis. Then ask them to draw a point on the graph for each pair of values in their input/output tables. (Since the output increases much faster than the input, it may be helpful to have two different scales on the horizontal and vertical axes. For example, if the horizontal scale goes from 0 to 15, the vertical scale should go from 0 to 225.)



Ask:

- ♦ *Do you see any pattern in the shape of the graph?*
- ♦ *Can you see where the next point would be if you plotted the values for the next larger square?*
- ♦ *Do the points line up on a straight line?*
- ♦ *If not, what kind of shape is formed by the points?*

- ♦ Ask students if it would make sense to join the points on the graph by a smooth curved line. Some students may say that it does not make sense to draw a smooth curve because every point on this graph represents a length and a number of squares, and none of the points in between represent such a square.

To help students make sense of this, ask if there is any way to make a toothpick square that contains 20 small squares, 40 small squares, etc. The answer is, of course not. The first would have to have a length between 4 and 5 toothpicks. The second would have to have a length between 6 and 7 toothpicks.

Explain that on the other hand, it may be helpful to draw a dotted line that represents the general trend of the data. This makes it easier to see the shape of the graph, make predictions, and so forth.

Part 2: How Many Toothpicks?

- 1 Ask students to look again at the toothpick squares and drawings they made in Part 1. Then ask them to make an input/output table with L , the length of one side of a large square, as input, and T , the total number of toothpicks needed to construct each square, as output.

Teaching Tip

The relationship between T and L in this activity is another quadratic expression, but it has a more complex form: $T = 2L^2 + 2L$. It is like the general quadratic relationship $y = ax^2 + bx + c$, but in this case $a = b = 2$, and $c = 0$. The curve of the graph will still be part of a parabola, but will increase faster than the curve of $S = L^2$.

Input: L, length of side of square	Output: T, number of toothpicks needed
1	4
2	12
3	24
4	40
5	60
6	
7	

- 2** Encourage students to look for patterns in the data. Ask them to predict the number of toothpicks needed to make a square with a side of length 6, length 7, length 10, length 15, and so on.

Ask students if they can predict how many *more* toothpicks will be needed for the *next* larger square. If necessary, have students continue constructing or drawing squares until they notice a pattern and are able to describe it.

- 3** Ask students to predict how much the number of toothpicks will increase from one square to the next. They should draw or construct the next square to test their predictions.

Students will notice a clear pattern: the number of toothpicks increases first by 8, then by 12, 16, 20, etc. Ask: *Does this increase represent a linear relationship?* [It does, because the number always increases by 4.]

Ask students to describe the relationship between I, the increase, and L, the length of the square. If necessary, they should make a table with L as the input and I as the output.

Since I is 8 when L is 1, 20 when L is 2, and so forth, students should work out the linear relationship $I = 4L + 4$. Therefore, if they know the number of toothpicks needed for a square of length L, then they have to have $4L + 4$ *more* toothpicks to build the next square. They should check that this relationship fits the data for the squares they have already built.

Teaching Tip

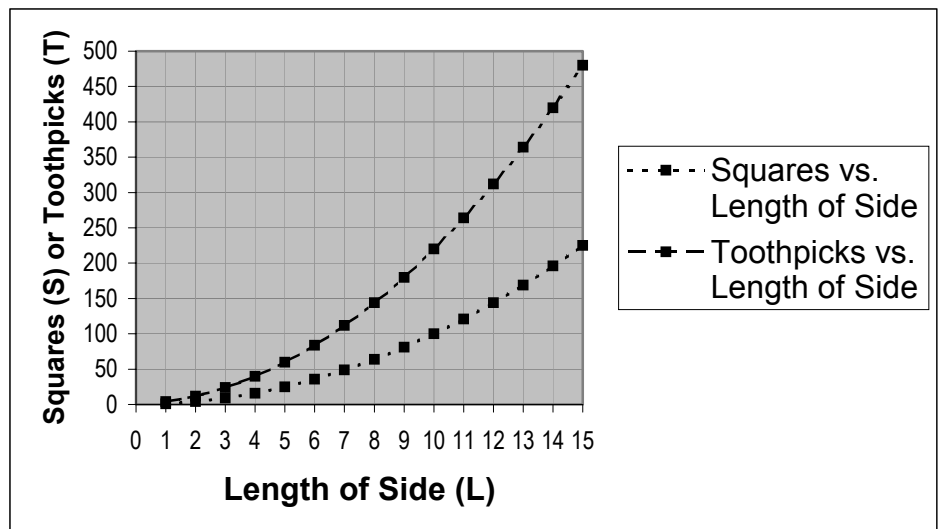
The fact that the amount of increase changes in a linear relationship is another feature of quadratic relationships.

- 4** Ask students to plot a graph, with L , the length, on the horizontal axis and T , the number of toothpicks needed, along the vertical axis. (If the scale on the horizontal axis goes from 0 to 10, the scale on the vertical axis will need to go from 0 to 220.)

Ask them to plot as many points as they can fit on their graphs. They do not need to build all the squares, just to use the number pattern they have observed to find values of T for values of L that go from 5 to 6 to 7 (etc.) to 10, to fill in the table, and to plot points on the graph.

Ask students if they should draw a smooth curve or a “dotted” curve through the points they have plotted. [The curve should be dotted because the points in between the points they have already plotted do not have any meaning in terms of toothpicks.]

Ask students to compare this curve with the curve they drew in Part 1. They should notice that the two curve in similar ways, but the curve of T vs. L is steeper (increases faster) than the curve of S vs. L .



- 5 Ask students if they can find a relationship between T and L; that is, if you give them any value of L, can they find the appropriate value of T? If students have completed the Linear Functions unit, you can ask them to think of this as a mystery machine. Ask them to try to find the rule for this mystery machine.

Teaching Tip

This is actually somewhat difficult. It is much more difficult to determine quadratic relationships than linear ones. One strategy is to factor the value of L from the output side of the table.

Input: L, length of side of square	Output: T, number of toothpicks needed
1	4 = 1 x 4
2	12 = 2 x 6
3	24 = 3 x 8
4	40 = 4 x 10
5	60 = 5 x 12
6	
7	

The advantage of this strategy is that students can see that the first factor is L; the second factor increases in a linear relationship. Ask students if they can find the rule for that relationship, that is, describe the relationship between the second factor and L.

The second factor increases by 2, and its value when L is 0 would be 2 (2 less than 4). Therefore, the linear relationship can be expressed as $2L + 2$. To get the value of T, you have to multiply $2L + 2$ by L, or $T = L(2L + 2)$. This can be rewritten as $T = 2L(L + 1)$ or $T = 2L^2 + 2L$.

A second strategy will work only if students know something about general quadratic relationships of the form $y = ax^2 + bx + c$ (or, in this case, $T = aL^2 + bL + c$) and about solving simultaneous equations. Then they can use the first three values of L and T to find equations for a, b and c, and use any method they can to solve those equations. By substituting values of L and T into the equation $T = aL^2 + bL + c$, students can get the following equations:

- ♦ If $L = 1$ and $T = 4$, then $4 = a + b + c$.
- ♦ If $L = 2$ and $T = 12$, then $12 = 4a + 2b + c$.
- ♦ If $L = 3$ and $T = 24$, then $24 = 9a + 3b + c$.

These equations can then be used to solve for a , b , and c . The result is $a = 2$, $b = 2$, and $c = 0$, which means $T = 2L^2 + 2L$.

Handshakes, Staircases, and Diagonals

Overview

Mathematical Focus

- ▶ Patterns in visual representations
- ▶ Arithmetical patterns that match visual patterns
- ▶ Rules for finding any element in a pattern, using both words and expressions
- ▶ Input/output tables and graphs as representations of patterns
- ▶ Visual representations of problem situations
- ▶ Quadratic relationships in the form $y = a(x^2 + bx)$ or $y = ax(x + b)$, where b may be positive or negative

In this activity, students think about three mathematical patterns that are very similar and can be described by similar quadratic formulas. This activity builds on the methods students used in Activity 1, Part 2, to find the relationship between the number of toothpicks and the length of a square.

Part 1, the handshake problem, is a classic mathematical puzzle. Suppose there are a number of people in a room, and they shake hands with every other person exactly once. The puzzle is, if there are n people in the room, how many handshakes will occur? The formula for this pattern is $y = 1 / 2(n)(n - 1)$ or $y = 1 / 2(n^2 - n)$.

The second puzzle arises from geometry. If you build a staircase with cubes that is n steps high, how many cubes will you need? The formula for this pattern is $y = 1 / 2(n)(n + 1)$ or $y = 1 / 2(n^2 + n)$.

The third puzzle also arises from geometry. If you have an n -sided polygon and connect all the vertices to each one another by diagonals, how many diagonals can you draw? The formula for this pattern is $y = 1 / 2(n)(n - 3)$ or $y = 1 / 2(n^2 - 3n)$.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Problem descriptions
- ▶ Graph paper
- ▶ Wooden cubes (optional)

Notes

These experiences build directly from the toothpick squares in Activity 1. You may need to help students recognize that an algebraic expression in the form $y = ax(x + b)$ is quadratic because if you multiply it out you get $y = a(x^2 + bx)$.

Activity

Part 1: How Many Handshakes?

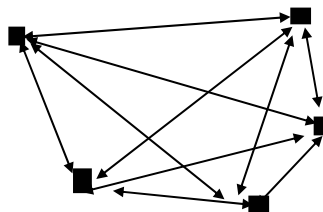
- 1** Pose this question to students: *Suppose you walk into a room and there are three people already there. You're the fourth person. If every person shakes hands once with everyone else, how many handshakes will take place?* Give students some time to think about it.
- ♦ Explain that the answer is six. One way to think about it is: I shake hands with three people. The next person shakes hands with two others (because he or she already shook hands with me.) The third person has only one person to shake hands with because that person has already shaken two people's hands. So $3 + 2 + 1 = 6$.
 - ♦ Ask students how many handshakes there would be if there were 5 people in the room, 6 people, or 10 people. Ask them if there is a rule that can tell how many handshakes will occur if you know the number of people.

After students have worked out the number of handshakes for 2, 3, and 4 people, ask them to complete the following table:

# of people, n	1	2	3	4	5	6	7	8	9	10
# of handshakes	0	1	3	6						

[The correct numbers of handshakes for 5–10 people are 10, 15, 21, 28, 36, and 45.]

Encourage students to draw diagrams, where each dot represents a person, and a line or an arrow connecting dots represents a handshake.



This drawing shows the number of lines (handshakes) there will be with 5 people in the room.

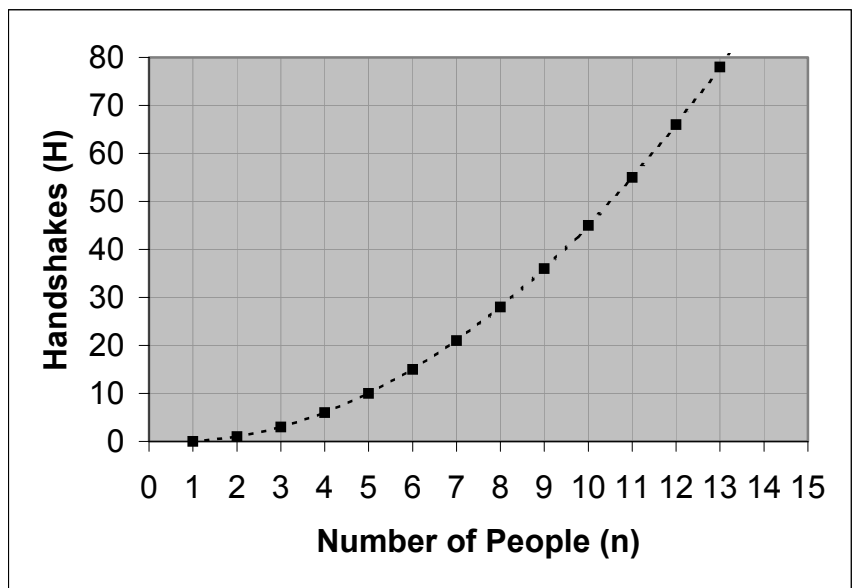
2 Now ask students to look for patterns. Students may notice several things:

- ♦ The number of handshakes increases by 1 each time.
- ♦ The number of handshakes for a particular number of people, n , can be found by adding all the numbers from 1 to $n - 1$.
- ♦ If you know the number of handshakes for a particular number, n , you can get the next number by adding n to the existing number of handshakes.

To see which of these ideas students might be using, ask students how they would figure out the number of handshakes needed for 11 people if they know the number of handshakes needed for 10 people. [One way is add 10 to the number of handshakes for 10 people (45) to get 55; another way is to add the numbers $1 + 2 + 3 + \dots + 10 = 55$.]

Ask how many handshakes are needed for n people, and ask whether the students can use the pattern to find how many are needed for $n + 1$ people. [Again, one possible answer is add n to the number of handshakes for n people. Another is to add all the integers from 1 to n .]

3 Suggest that students think of the data they have been gathering as an input/output table. Ask them to make a graph, plotting n , the number of people, on the horizontal axis, and H , the number of handshakes, on the vertical axis. Have them connect the points with a dotted line. Ask them to describe the graph. [It's a smooth curve that gets steeper as n gets bigger.]



4 Ask students if they can find a rule that will give them H , the number of handshakes, if they know “ n ”, from the number of people. [The rule is $H = n(n-1)/2$ or $n^2/2 - n/2$.] There are several ways that students may be able to find this rule:

a. Students can use the diagram, shown above in number 1, or use reasoning about people and handshakes. They should first figure out how many lines are connected to each person ($n-1$) then think about how many people are in the room (n). Next, they should multiply $n \times (n-1)$ to get the total number of handshakes. However, this will count every handshake twice; i.e., if I shake hands with you, I don’t count you shaking hands with me as a separate handshake. So $n(n-)$ has to be divided by 2 to get the total, $n(n-1)/2$.

b. Students can rewrite the number pattern for the number of handshakes as $n \times$ some number. Here’s what the table will look like:

# of people, n	1	2	3	4	5	6	7	8	9	10	n
# of handshakes	0 = 1 x 0	1 = 2 x 1/2	3 = 3 x 1	6 = 4 x 3/2	10 = 5 x 2	15 = 6 x 5/2	21 = 7 x 3	28 = 8 x 7/2	36 = 9 x 4	45 = 10 x 9/2	$n \times$ $(n-1)/2$

If students examine the numbers n is multiplied by, they’ll see that the numbers increase by $1/2$, and that they form a linear sequence, 0, $1/2$, 1, $3/2$, 2, etc., which can be described as $(n-1)/2$. So the number of handshakes for n people is the product of $n(n-1)/2$.

c. To add the numbers $1 + 2 + 3 + \dots + (n - 1)$, students can reverse the series and add it to itself:

$$\begin{array}{r}
 1 + 2 + 3 + \dots + (n - 1) \\
 + (n - 1) + (n - 2) + (n - 3) + \dots + 1 \\
 \hline
 n + n + n + \dots + n
 \end{array}$$

Each partial sum is equal to n , and there are $n - 1$ of them, so that the sum of the double sequence is $n(n - 1)$. That sum has to be divided by 2: $n(n - 1)/2$.

5 Ask students to tell you the total number of handshakes needed for 20 people, 100, 1,000, etc. As an extra challenge, ask students to determine the following:

- ♦ If there were 190 handshakes, how many people were in the room? [20 people] 210 handshakes? [21 people] 2,450? [100 people] 2,550? [101 people].
- ♦ Is it possible to have 200 handshakes? How about 400? Why or why not?

[Neither of these numbers can be a number of handshakes because there are no integers for which $n(n - 1)/2 = 200$ or 400 .]

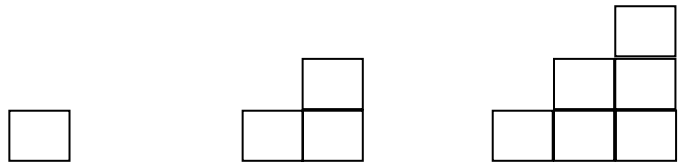
Ask students to make up two “How many people?” puzzles — one that has a valid solution and one that does not.

Part 2: How Many Cubes in a Staircase?

Teaching Tip

An interesting feature of this problem is that the number sequence, 1, 3, 6, 10, ... is the same as the sequence for the handshake problem. The only difference is where it starts. For the handshake problem, the sequence starts when $n = 2$ (i.e., two people are needed for one handshake). For the staircase problem, the sequence starts when $n = 1$ (i.e., one block is needed for the first staircase). You may ask students to think about the similarities and differences in the sequences, and the mathematical rules, after they have completed Part 2.

- 1** Ask students to build or draw a sequence of staircases built from cubes:



Ask them to fill in the following table:

n, # of steps	1	2	3	4	5	6	7	8	9	10
C, # of cubes	1	3	6							

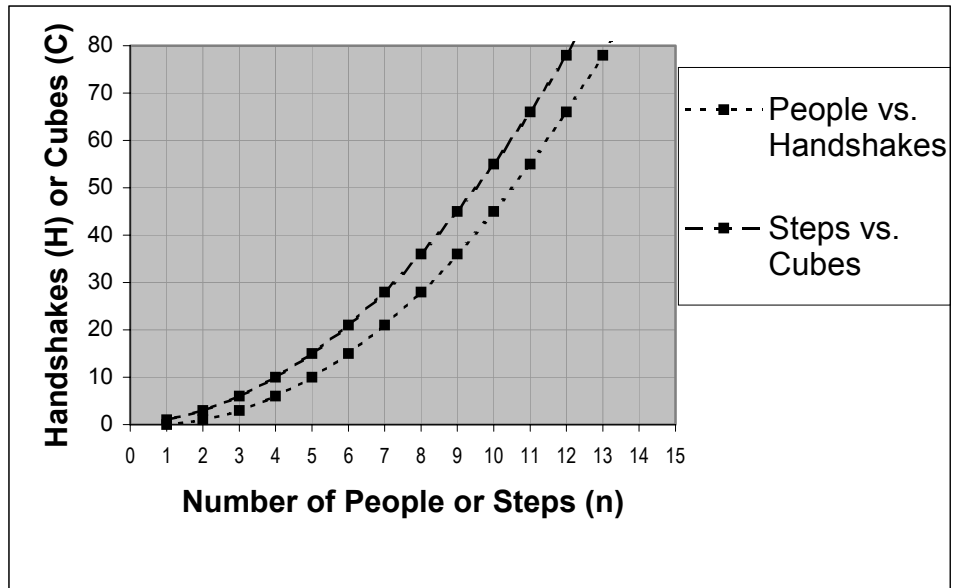
- 2** Ask students what patterns they see. Students might notice several patterns:

- ♦ The number of cubes increases by 2, 3, 4, 5 ... 1 *more* each time the number of steps increases by 1.
- ♦ To go from the n th value of C to the next value of C , you have to add $n + 1$ to the current value of C .
- ♦ Students should recognize that the increasing pattern for the number of cubes is almost the same as the pattern for the number of handshakes. The difference is that the handshake pattern starts with 1 handshake when $n = 2$; the staircase pattern starts with 1 cube when $n = 1$. (With the handshake pattern, n was added to the number of handshakes to get the next value; with the staircase, $n + 1$ is added.)

Ask students if they know the number of cubes in a 10-step staircase and in an 11-step staircase. [55 in a 10-step staircase and 66 in an 11-step staircase]

- 3** Ask students to take out the graph they made for Part 1, and plot the graph for the staircase problem on the same axes: n along the horizontal axis and C along the vertical axis. After plotting the points, have students connect them with a dotted line.

Ask students to describe the curve and compare it to the curve they drew for the handshake problem. [They should describe a smooth curve that gets steeper as n gets bigger. The staircase curve should look the same as the handshake curve, shifted one unit to the left on the graph paper (it passes through 1, 1 while the graph for the handshake pattern passes through 2, 1).]



4 Ask students to try to find the relationship between n and C . They can use any of the methods they used for the handshake problem. In addition, they can use drawings. Here are some possible strategies:

a. If students start from the table and use the factoring approach, their table will look like this:

n, # of steps	1	2	3	4	5	6	7	8	9	10	n
C, # of cubes needed	1 = 1 x 1	3 = 2 x 3/2	6 = 3 x 2	10 = 4 x 5/2	15 = 5 x 3	21 = 6 x 7/2	28 = 7 x 4	36 = 8 x 9/2	45 = 9 x 5	55 = 10 x 11/2	$n \times (n + 1)/2$

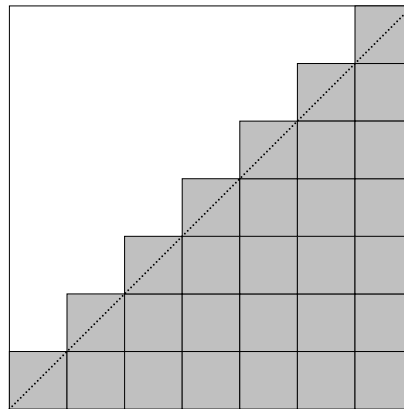
If students examine the numbers n is multiplied by, they'll see that the numbers increase by $1/2$, in the pattern 1, $3/2$, 2, $5/2$, etc., and that for the n th staircase, the factor can be written as $(n + 1)/2$. So the number of cubes needed for an n -step staircase is the product of $n(n + 1)/2$.

b. To add the numbers $1 + 2 + 3 + \dots + n$, students can reverse the series and add it to itself:

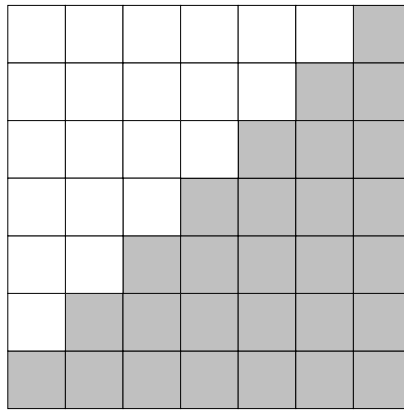
$$\begin{array}{r} 1 + 2 + 3 + \dots + n \\ + \quad n + (n-1) + (n-2) + \dots + 1 \\ \hline (n+1) + (n+1) + (n+1) + \dots + (n+1) \end{array}$$

Each partial sum is equal to $n + 1$, and there are n of them, so the double sum is $n(n + 1)$. The final result has to be divided by 2: $n(n + 1)/2$.

c. Ask students to see if they can use either a (1) drawing of an n -step staircase within an n by n square or (2) a drawing of two n -step staircases, combined into a rectangle, to find a relationship between n and the number of cubes.



The total number of cubes in the square (the area of the square) is $n \times n$, or n^2 . The dotted line divides the area in half, so that the shaded portion represents $n^2/2$. However, $1/2$ of each cube along and above the diagonal line is also shaded and has to be added. There are n cubes along the diagonal, so the total number of shaded cubes is $n^2/2 + n/2$.



The two staircases fit together to make a rectangle. The rectangle is n cubes high and $n + 1$ cubes long. The total number of cubes in the rectangle is $n(n + 1)$. However, one staircase is only half the rectangle, so the number of cubes in one staircase is $n(n + 1)/2$. This can be rewritten as $(n^2 + n)/2$, or $n^2/2 + n/2$.

5 Ask students to tell you the total number of cubes needed for a 20-step staircase, for a 30-step staircase, 100 step, 1,000 step, etc.

6 Challenge students with questions such as the following:

- ♦ If 190 cubes were used, how many steps were in the staircase? [19 steps] 210 cubes? [20 steps] 2,450? [99 steps] 2,550? [100 steps]
- ♦ Is it possible to have a staircase that uses 200 cubes? How about 400? Why or why not?

[Neither of these numbers can be the number of cubes, because there are no integers for which $n(n + 1)/2 = 200$ or 400 .]

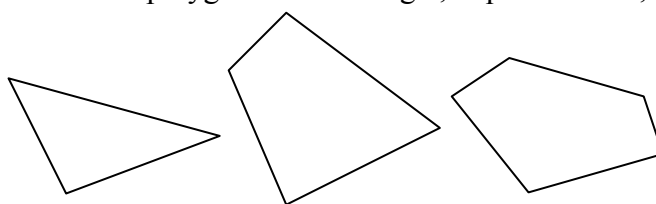
7 Ask students to make up two “How many cubes?” puzzles — one that has a valid solution and one that does not.

Part 3: How Many Diagonals Can Be Drawn in an n-Sided Polygon?

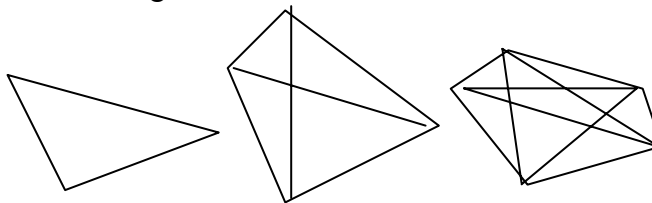
Teaching
Tip

The “how many diagonals?” problem is similar to both the handshake problem and the staircase problem. Each term in the number sequence for this problem, i.e. 0, 2, 4, 9, 14, etc., is one less than the corresponding term in the two previous problems (i.e., 1, 3, 6, 10, 15, etc.). The sequence does not start until $n = 3$ (the smallest possible polygon is a triangle, and a triangle does not have any diagonals). This is why the term $(n - 3)$ appears in the formula, rather than $(n - 1)$ as in the handshake problem or $(n + 1)$ as in the staircase problem. Part of the activity that follows asks students to compare the tables, mathematical rules, and graphs they used for the handshake problem and the staircase problem with those they use for the “how many diagonals” problem.

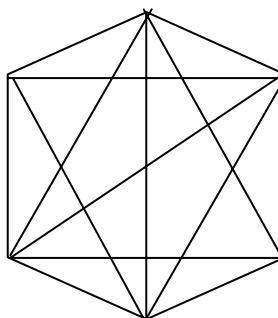
- 1 Draw three polygons like the ones shown below, and tell students that these polygons are a triangle, a quadrilateral, and a pentagon.



Give students some paper and ask them to draw a 3-sided polygon, a 4-sided polygon, and a 5-sided polygon, similar to the ones shown below, and to draw in all the diagonals for each polygon. Their drawings should look like this:



Have students draw a hexagon ($n = 6$) with a complete set of diagonals. Make sure that they join each vertex to every other vertex of the hexagon.



Then have students make up a table to record their results, showing both the number of diagonals connected to each vertex and the total number of diagonals for each polygon.

Sides, n	3	4	5	6	7	8	9	10	n
Diagonals Connected to Each Vertex	0	1	2	3	4	5	6	7	$n - 3$
Total Number of Diagonals, d	0	2	5	9	14	20	27	35	

2

Ask students to look at the second row of the table and describe any patterns they see in the way the number of diagonals connected to each vertex changes as the number of sides increases. [The number of diagonals at a vertex increases by 1 each time the number of sides increases by 1.] Ask students to verify this by drawing a seven-sided polygon (a heptagon) and seeing how many diagonals it has at each vertex.

Ask students if they think this relationship is linear. Students should recognize that it is because the number of diagonals at each vertex increases by a constant, 1, with every new polygon. Students may say that the number of diagonals at each vertex of an n -sided polygon is $n - 3$. (Another way to think of this is to recognize that diagonals can be drawn to all vertices except the two adjacent vertices and the starting point. So if a polygon has n vertices, $n - 3$ diagonals can be drawn from each one.)

Now ask students to look at the third row of the table and describe any patterns they see relating to the total number of diagonals in a polygon. They may see some or all of the following patterns:

- ♦ The number of diagonals increases by 2, then 3, then 4, and so on — 1 more each time.
- ♦ To go from the diagonals in an n -sided polygon to an $(n + 1)$ -sided polygon, you add $n - 1$ more diagonals (from 3-sided to 4-sided, add 2; from 4-sided to 5-sided, add 3; from 5-sided to 6-sided, add 4; and so on.)

Ask students whether they think this relationship is linear. [It is not, because the amount of increase is not constant.]

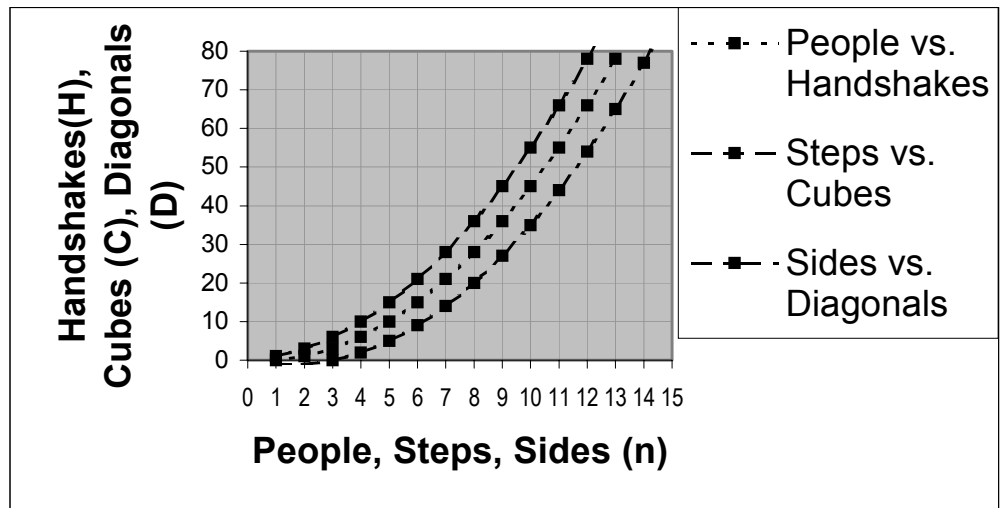
Ask students to compare the number pattern for the number of diagonals with the number patterns for the handshakes and staircases:

Input (n)	1	2	3	4	5	6	7	8	9	10
Diagonals			0	2	5	9	14	20	27	35
Handshakes		1	3	6	10	15	21	28	36	
Cubes in a staircase	1	3	6	10	15	21	28	36		

Students may recognize that each number in the newest pattern can be gotten by subtracting 1 from the corresponding number in the earlier patterns, and that the new sequence starts with 0 when $n = 3$.

3

Ask students to plot a graph of n (number of sides) and d (total number of diagonals). Have them draw it on the same graph as they used for Parts 1 and 2 of this activity. After plotting the points, have students connect them with a faint line. Ask them to describe the curve and how it relates to each of the other curves. [It seems to curve in exactly the same way, but it's shifted to the right and lower than the other curves.]



4 Tell students that it's time to find the relationship between n and d . Explain that just as with the earlier examples, there is more than one way to do this:

a. Probably the most direct way is to start from the fact that there are $n - 3$ diagonals at any vertex. Since there are n vertices, you need to multiply $n(n - 3)$ to get a total. However, this total counts every diagonal twice (the diagonal from A to B is the same as the diagonal from B to A), so you have to divide by 2 to get the real total: $d = n(n - 3)/2$.

b. A second method is to go back to the table of numbers and rewrite each value in the table as n multiplied by another number:

Once again, n is multiplied by linear increases of $1/2$: 0, $1/2$, 1, $3/2$, 2, etc. By comparing the three rows of the table, it's easy to see that $d = n(n - 3)/2$.

Sides, n	3	4	5	6	7	8	9	10	n
Diagonals Connected to Each Vertex	0	1	2	3	4	5	6	7	$n-3$
Total Number of Diagonals, d	0 (3x0)	2 (4x1/2)	5 (5x1)	9 (6x3/2)	14 (7x2)	20 (8x5/2)	27 (9x3)	35 (10x7/2)	$n \times$ $(n-2)/2$

5 Ask students to tell you the total number of diagonals for a 20-gon, a 30-gon, a 100-gon, a 1,000-gon, etc.

6 Ask students some questions such as:

- ♦ If a polygon has 170 diagonals, how many vertices must it have? [20 vertices] 189 diagonals? [21 vertices] 4850? [100 vertices] 4949? [101 vertices]
- ♦ Is it possible to have a polygon with 200 diagonals? How about 400? Why or why not?

[Neither of these numbers can be the number of diagonals of a polygon, because there are no integers for which $n(n-3)/2 = 200$ or 400.]

Ask students to make up two “How many diagonals?” puzzles — make up one that has a valid solution and one that does not.

Cube Patterns

3

Overview

Mathematical Focus

- ▶ Patterns in visual representations
- ▶ Matches between arithmetical patterns and visual patterns
- ▶ Rules for finding any element in a pattern, using both words and expressions
- ▶ Input/output tables and graphs as representations of patterns
- ▶ Cubic relationships in the form $y = x^3$ and $y = ax^2(x + b)$

In this activity, students build a series of cubes and discover and graph a cubic relationship. They then construct a series of cubes with the tops cut off and discover and graph a relationship for that pattern.

Preparation and Materials

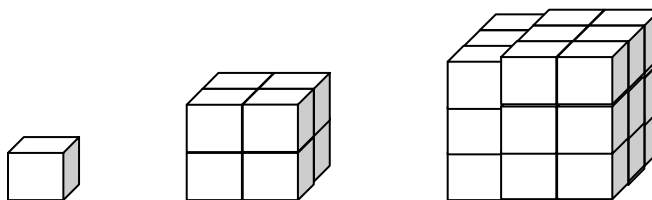
Before the session, gather the following materials:

- ▶ Problem descriptions
- ▶ A large number of small cubes
- ▶ Graph paper

Activity

Part 1: Building a Series of Cubes

- 1** Ask students to construct a series of cubes, starting with just 1 block on a side, then 2 blocks on a side, then 3, and so on. You may want to label these a 1-cube, 2-cube, 3-cube, etc. As students build, ask them to record the number of blocks they use in a table:



Length of an edge (n)	1	2	3	4	5	6
# of blocks needed (B)	1	8	27			

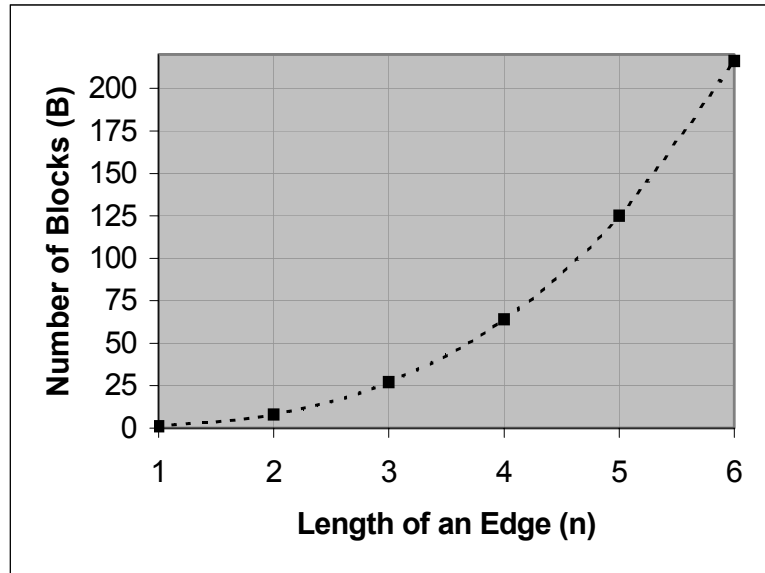
After building three cubes in the sequence, ask students if they can predict the number of blocks needed to build the next larger cube in the pattern. Ask students if they can see a pattern in how the number of blocks changes from one large cube to the next.

(Unless students realize that the number of blocks is equal to the “cube” of the number of cubes on a side [that number multiplied by itself 3 times], it may be difficult for them to predict the next term in the sequence.)

Ask students to build the 4-cube, 4 blocks on a side, and to fill in the next value on the table [64 blocks]. Ask if they can predict how many blocks will be needed for a 5-cube.

Some students may now realize that the number of blocks is equal to n^3 , where n is the number of blocks along one edge. If not, ask them to count the blocks in each layer as they build. The 2-cube has 2 layers of 4 blocks each; the 3-cube has 3 layers of 9 blocks each; and so forth. Since the large cube is n blocks high, and each layer contains a square of $n \times n$ blocks, students should be able to predict that the total number of blocks needed for an n -cube is $n \times n \times n$, or n^3 .

- 2** Ask students to make a graph of B vs. n . (The vertical [B] axis will need to go from 0 to over 200, just to accommodate the number of blocks needed for a 5-cube and a 6-cube.) Ask them to describe the graph.

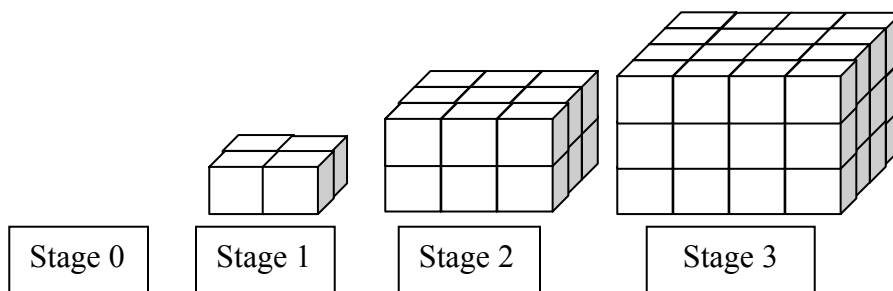


Students should observe that this graph is much steeper than for a quadratic relationship, that is, the values along the vertical axis grow much more rapidly.

- 3** Ask students to predict the number of blocks needed to build a 20-cube and a 100-cube. [8,000 and 1,000,000 blocks, respectively]
- ♦ Ask students which cube will need 1,000 small blocks to build it, which will need 1,728 blocks, and which will need 3,375. [a 10-cube, a 12-cube, and a 15-cube]
 - ♦ Ask whether students could build a large cube from exactly 50 blocks, 100 blocks, or 200 blocks. [Answer is no, because there are no integers whose cubes are 50, 100, or 200.]

Part 2: A Truncated Cube

- 1** Ask students to consider this pattern of shapes built from cubes, thinking of the pattern as a set of rectangular prisms with square bases:



- 2** Ask students to make a table showing the number of blocks needed to build each stage:

Stage, n	1	2	3	4	5	6
# of blocks, B	4	18	48			

After students have built the construction for Stage 3— a prism with a 4×4 base— ask them to predict the number of blocks needed for stage 4. [100]

- 3** Ask students if they can write an equation for the number of blocks, B , in Stage n .

- One way to do this is to multiply the length by the width by the height. The base uses $(n + 1) \times (n + 1)$ blocks, and there are n layers of blocks. Therefore, the equation is $B = (n + 1)(n + 1)(n)$, which is equivalent to $(n)(n^2 + 2n + 1)$ or $n^3 + 2n^2 + n$.
- Another way is to think about the construction as a large cube using $(n + 1)^3$ blocks, with one layer of blocks missing, for a total of $(n + 1)^3 - (n + 1)^2$, which simplifies to $n^3 + 2n^2 + n$.

4

Tell students that there's another way to predict the behavior of a sequence of numerical terms, if it's consistent. It has to do with determining the differences from one term to the next, then taking the differences of the differences, and so forth, to see if a pattern emerges.

To illustrate, make sure that students have constructed five stages of the geometric pattern above. Their results up to this point are as follows:

Stage (n)	1	2	3	4	5	6
# of blocks (B)	4	18	48	100	180	

It should be obvious to students that the number of blocks is increasing rapidly, but it may be difficult for them to predict the next term without building it or determining the formula. Have them look at the differences from term to term:

Pattern:	4	18	48	100	180
1 st differences		14	30	52	80
2 nd differences			16	22	28
3 rd differences				6	6

Explain that this pattern allows them to continue the original sequence as far as they'd like. For example, the bold terms show how to find the next term in the original sequence:

Pattern:	4	18	48	100	180	294
1 st differences		14	30	52	80	114
2 nd differences			16	22	28	34
3 rd differences				6	6	6

The next term in the second difference pattern is $28 + 6$, or 34; the next term in the first difference pattern is $80 + 34$, or 114; and the next term in the original pattern is $180 + 114$, or 294. This is equal to what you get from the formula for $n = 6$: $(7 \times 7) \times (6) = 49 \times 6 = 294$.

Suggest that students use this method of looking for differences on the construction of large cubes in Part 1, and on some of the constructions Activity 2.

The Cube Painting Problem

Overview

Mathematical Focus

- ▶ Patterns in visual representations
- ▶ Matches between arithmetical patterns and visual patterns
- ▶ Rules for finding any element in a pattern, using both words and expressions
- ▶ Input/output tables and graphs as representations of patterns
- ▶ Compare constant relationships, linear, quadratic, and cubic relationships in the same problem contexts

In this activity, students imagine the process of assembling a series of large cubes, like the ones in Activity 3. However, this time, the outer surface of the large cube—and only the outer surface—is to be painted. Their task is to determine for the n th cube, which is made up of $n \times n \times n$ small cubes, the number of cubes that are painted on 3 adjacent faces, on 2 adjacent faces, and on just 1 face, as well as the number of unpainted cubes.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Problem descriptions
- ▶ A large number of blank cubes
- ▶ Graph paper
- ▶ Small colored stickers

Activity

1 There are a number of ways to pose the cube painting problem. Choose one of the two versions presented here and read it to students or let them read it to themselves. Here's one: You are manager of a puzzle factory. One of your most popular puzzles is the painted cube puzzle: a large cube made of smaller cubic blocks. Each of the blocks is painted on one side, two sides, or three adjacent sides, or not painted at all. When the puzzle is assembled properly, the outside of the large cube is painted—and only the outside.

Your puzzle factory is committed to sending out different size puzzles, for example, a 4-cube ($4 \times 4 \times 4$), an 8-cube, or even a 10-cube. You promise to send out any size puzzle that someone wishes to order (for a price, of course). In order to do this, you need to know how many of each type of painted block are needed for any possible cube.

Another way to think of the problem is this: Suppose you build an n -cube from small cubic blocks. You glue all the blocks together lightly and drop the entire cube into a vat of paint, so that the entire surface is painted. Then you remove the cube from the paint, let it dry, and separate all the glued blocks.

2 Ask students to build a 2-cube, a 3-cube, and a 4-cube. For each cube, they should simulate painting by putting a sticker on the outside of every small block. Then they should count the number of each type of “painted block” and record the results in a large chart, such as this:

Size of Cube	Total # of Blocks	Blocks Painted on 3 Surfaces	Blocks Painted on 2 Surfaces	Blocks Painted on 1 Surface	Unpainted Blocks
2-cube	8				
3-cube	27				
4-cube	64				
5-cube					
6-cube					
7-cube					
8-cube					
9-cube					
10-cube					
n-cube	n^3	8	$12(n - 2)$	$6(n - 2)^2$	$(n - 2)^3$

As students fill in the chart, ask them to make predictions about how many of each type of block they will need for a 5-cube, a 6-cube, and so forth.

Students may use a variety of different strategies to approach the problem:

- ♦ Some will build as many large cubes as they can, until they are sure of the pattern.
- ♦ Some will draw pictures of one face of each large cube and use that to reason about what they might need for a 3-dimensional cube.
- ♦ Some may build parts of a cube: just the edges, or just one face, or three adjacent edges.

3 As students build or draw their cubes, ask the following questions:

- ♦ *How many faces does a cube have?* [6] *How many edges?* [12: 4 on top, 4 on bottom, and 4 around the sides] *How many corners?*[8]
- ♦ *Where on each cube do you find blocks with three surfaces painted?* [At each corner]
- ♦ *Where do you find two surfaces painted?*[Along each edge, not including the corners]
- ♦ *Where do you find one surface painted?* [On a square inside each face, not on the edges or corners]
- ♦ *Where do you find unpainted blocks?* [Inside each cube]

Thinking about this may be enough for some students to discover the formulas for each type of block, which are as follows:

- ♦ For 3 surfaces painted, the number of blocks is constant: 8, no matter how large the cube is.
- ♦ For 2 surfaces painted, there are 12 edges \times the length of each edge after subtracting the corners; the number is $12 \times (n - 2)$.
- ♦ For 1 surface painted, the blocks are arranged into a square on each face, and each square is $n - 2$ blocks long on each side; the number is $6 \times (n - 2)^2$.
- ♦ For 0 surfaces painted, the surfaces inside the cube are like another cube, with 2 fewer blocks in each direction than the larger cube; the number is $(n - 2)^3$.

4 Some students may have reasoned out the formulas as described above. Others will need to use the number patterns obtained by drawing or building several cubes and examining the number patterns that result. Eventually, their charts should be completed as follows:

Size of Cube	Total # of Blocks	Blocks Painted on 3 Surfaces	Blocks Painted on 2 Surfaces	Blocks Painted on 1 Surface	Unpainted Blocks
2-cube	8	8	0	0	0
3-cube	27	8	12	6	1
4-cube	64	8	24	24	8
5-cube	125	8	36	54	27
6-cube	216	8	48	96	64
7-cube	343	8	60	150	125
8-cube	512	8	72	216	216
9-cube	729	8	84	294	343
10-cube	1,000	8	96	384	512
n-cube	n^3	8	$12(n - 2)$	$6(n - 2)^2$	$(n - 2)^3$

5 Ask students to draw a graph (on one set of axes) plotting as many points from the table as fit on a page.

- 6** Ask students if they can tell you what each type of relationship is:
- ♦ The number of 3-surface blocks [A constant, 8]
 - ♦ The number of 2-surface blocks [A linear relationship, $12 \times (n - 2)$]
 - ♦ The number of 1-surface blocks [A quadratic relationship, $6 \times (n - 2)^2$]
 - ♦ The number of unpainted blocks [A cubic relationship, $(n - 2)^3$]

How Long Does It Take? (Speed and Time Problems)

Overview

Mathematical Focus

- ▶ Input/output tables and graphs as representations of patterns
- ▶ Rules for finding any element in a pattern, using both words and expressions
- ▶ Inverse relationships in the form $xy = a$ or $y = a/x$, where a is a constant

In this activity, students investigate examples of inverse relationships for which the product of the variables is a constant. Examples include speed and time to travel a fixed distance; numbers of workers and time to complete a job; and length and width of a rectangle with fixed area.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Problem descriptions
- ▶ Graph paper
- ▶ Hand calculator

Activity

Part 1: How long does it take to travel 1000 miles?

1 Ask students how long it would take to travel 1,000 miles.

Ask students to imagine traveling in at least eight different ways, real or imaginary. Some should be fast (a jet plane or rocket ship), some slow (walking or riding a donkey), and some in between (on horseback; in a stagecoach, car, or train).

For each method of travel, have students estimate a speed in miles per hour. Once they have done that they should figure out how long it will take (in hours) to travel 1,000 miles at that speed. Encourage students to guess if they can't estimate; for this problem, it doesn't matter if the speeds are accurate.

If students have difficulty figuring out the time, suggest that they start with some speeds that are simple factors of 1,000: 500 mph, 200 mph, 100 mph, 50 mph, 20 mph, and so forth. Ask them how long it would take to go 1,000 miles at each speed. Suggest that they work it out hour by hour: *At 200 mph, how far would you go in 1 hour? 2 hours? 3 hours?* And so forth, until 1,000 miles are reached.

Students should then record their results in a chart like this (the first few are just simple examples):

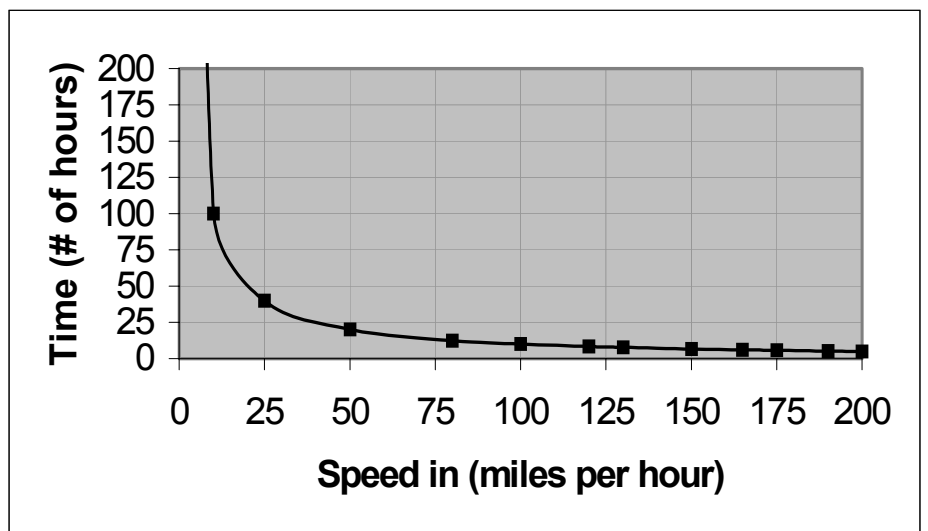
Method of Transportation	Speed	Travel Time
Jet plane	500 mph	2 hours
Fast train	90 mph	11 and 1/9 hours
Rocket	10,000 mph	1/10 hour
Stagecoach	15	66 and 2/3 hours

2 Once students have chosen 8–10 different means of travel, and calculated the time it would take to travel 1,000 miles, ask them if they notice any patterns in the data. [They should observe that the

product of the speed and time is always 1,000. This is a variation of the old formula, $\text{Rate} \times \text{Time} = \text{Distance}$.]

They may also observe that certain values give “opposite” results. For example, at 20 miles per hour, it would take 50 hours to travel 1,000 miles; at 50 mph, it would take 20 hours.

Ask students to make a graph of time vs. speed, with speed on the vertical axis and time on the horizontal axis. Before they draw the graph, ask students to predict the shape of the graph and draw their prediction. (Many students who recognize that time increases as speed decreases will draw a straight line sloping downward to the right, assuming a linear variation.)



- 3 After students have drawn their graphs, ask them to use the graph to estimate the time it would take to travel 1,000 miles at 300 mph (or some other speed they have not yet calculated).

Teaching Tip

To do this, students need to locate 300 mph on the vertical axis, make a horizontal line to meet the curve, then draw a vertical line downward. The value directly below the point on the curve gives the time estimate. For example, at 300 mph, the graph should show a time of $3\frac{1}{3}$ hours. Students may estimate this reading to be $3\frac{1}{2}$ or $3\frac{1}{4}$ hours.

- ♦ *What point could you draw on the graph if the speed was 1,000 mph, 2,000, 10,000, etc? Does this point fit on the graph? If you have a huge graph, where would this point appear? Can you plot it on your graph? Why or why not?*
- ♦ *Where on the graph could you mark the time for a speed of 1 mph, 0.5 mph, 0.1 mph, etc? Can you plot these points on your graph?*

Students should recognize that as speeds get very fast, the time gets very short; as speeds get very slow, the time grows very large. However, though the time will get very small at high speeds, it will never be 0. Similarly, as the speed decreases, the time takes longer and longer, but the speed will never be 0.

Point out to students that the speed/time problem is an example of an *inversely proportional relationship*, that is, a relationship between two variables for which the product of the variables is a constant.

Part 2: Area, Length, and Width

- 1 Ask students to draw a rectangle on graph paper with an area of 36, and have them record its length and width.

Next, ask them to draw as many different rectangles as they can, each with an area of 36. Ask them to record the lengths and widths in a table. A few examples are shown:

Length	Width
6	6
9	4
4	9

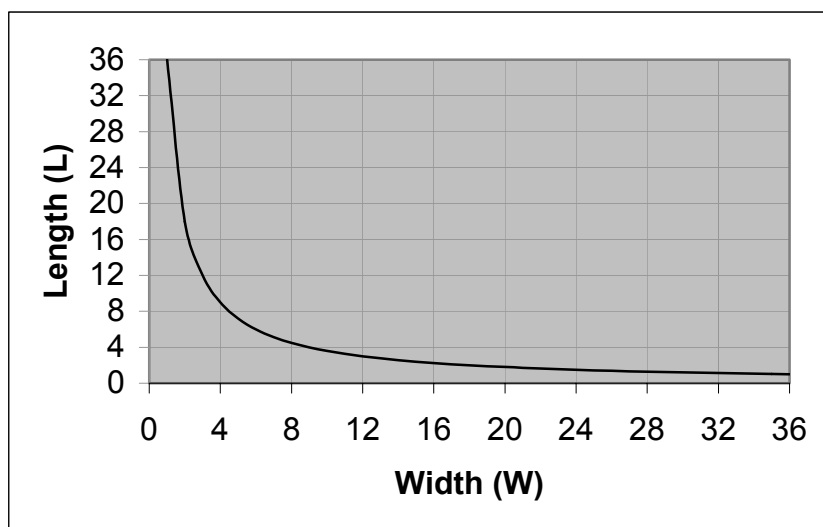
Ask students if they think they have found all the possible rectangles with an area of 36 square units. Tell them that there are 9 rectangles with integer lengths: 1×36 ; 2×18 ; 3×12 ; 4×9 ; 6×6 ; 9×4 ; 12×3 ; 18×2 ; and 36×1 .

Check to see if students notice any patterns among their results. (Some possible patterns are [1] when length gets larger, width gets smaller, and [2] rectangles come in pairs. (i.e., for every rectangle with length a and width b , there is another rectangle with length b and width a .)

Ask students what other rectangles they could draw. Have them draw a rectangle for which the length and/or width is not an integer. [Some examples are 8×4.5 and 4.5×8 ; 10×3.6 and 3.6×10 —and there are infinitely many more!]

Ask them if there are rectangles that have an area of 36 square units that cannot easily be drawn on graph paper. [Some examples are 0.5×72 , and 72×0.5 —and there are infinitely many more long skinny rectangles.]

- 2** Ask students to make a graph of length and width, using the values in their tables.



Once the graph is drawn, ask students to use the graph to estimate length-width pairs that they did not already use. For example, with length 5, the width is 7.2 (students will estimate anywhere from 7 to 7.25).

Ask students where the following values would appear on the graph: lengths of $1/2$, $1/4$, and $1/10$. [The widths would be way off the left edge: 72, 144 and 360, respectively] Next ask them where the lengths of 100, 200, or 300 would appear. [Way off the top of the graph]

Ask students to describe the longest, skinniest rectangle they can imagine whose area is 36. Ask them to imagine a rectangle that is 0.000001 units long and 36,000,000 units wide. Explain that it's hard to imagine, but if you had such a rectangle, its area would be exactly 36 square units.

Explain to students that the two problems they worked with in this activity are similar. Both are examples of inverse relationships in which the product of two variables is a constant. Both have the characteristic that one variable can get very large while the other gets very small—but neither variable can ever have 0 as a value. They can come very close to zero by letting the other variable get very big, but, of course, their product will always have the same constant value, no matter how small one gets and how large the other gets.

Part 3: Imagining Other Situations Described by Inverse Relationships

Ask students if they can imagine other situations where the product of two variables is a constant. Let them pick one of these situations, make up values for both variables, and then graph them.

You can suggest examples if students cannot make up their own:

- ♦ The time it takes to do a job (e.g., mow a lawn or paint a house), in relation to the number of people working on that job
- ♦ The number of steps it takes to walk 100 feet, in relation to the size of each step (assuming that step sizes remain the same for any particular walk)
- ♦ The number of identical toys you can buy for \$20, in relation to the price of each toy

Exponential Growth and Decay

Overview

Mathematical Focus

- ▶ Patterns in concrete situations
- ▶ Rules for finding any element in a pattern, using both words and expressions
- ▶ Input/output tables and graphs as representations of patterns
- ▶ Exponential relationships of the form $y = a^x$, where a is a constant

In this activity, students work with paper cutting to explore relationships that increase and decrease exponentially. For example, if you repeatedly cut a piece of paper in half, then cut the halves in half, and so on, the number of pieces increases from 1 to 2 to 4 to 8, while the area of each piece decreases by $1/2$, then $1/4$, then $1/8$, and so forth. This problem is used to create models of exponential growth and exponential decay. Other exponential growth/decay problems are compared to the paper-cutting problem. Exponential growth is also compared to linear and quadratic growth.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Student Page 1: 8 x 8 Grid, a few copies
- ▶ Student Page 2: 9 x 3 Grid, a few copies
- ▶ Scissors
- ▶ Graph paper

Activity

Teaching Tip

Exponential functions are of the form $y = a^{+/-x}$, where a is a constant and x varies. If x is positive, the function increases rapidly. If x is negative, $y = a^{-x}$ is the same as $y = 1/a^x$, which decreases, getting closer and closer to zero (which is also called reaching an asymptote at $y = 0$) as x increases. If x has only positive integer values, the terms of the positive function are $1, a, a^2, a^3, a^4$, etc. If x has only negative integer values, the terms are $1, 1/a, 1/a^2, 1/a^3, 1/a^4$, etc.

Part 1: How Many Pieces of Paper?

- 1 Ask students to cut out the 8 x 8 grid on Student Page 1. Explain that they are going to cut it in half, then cut each new piece in half, and continue doing that until they can determine a pattern for the number of pieces. Ask students to make an input/output table with the number of cuts as the input and the number of pieces as the output.

Number of Cuts	Number of pieces
0	1
1	2
2	
3	
4	

Once students have made a few cuts, ask if they can predict what will happen next. They should be able to say that the total number of pieces will double each time. Ask them to figure out—without cutting—how many pieces they will have after the tenth cut, the twentieth, and the n th cut.

Students should soon be able to work out that the number of pieces is equal to 2 multiplied by itself n times, that is, for $n = 1, 2 \times 1$; for $n = 2, 2 \times 2$; for $n = 3, 2 \times 2 \times 2$; for $n = n, 2 \times 2 \times \dots \times 2$ (n times).

- 2 If students don't bring up the idea of exponential notation, remind them that they know a short way to write 2×2 as 2^2 , $2 \times 2 \times 2$ as 2^3 , and $2 \times 2 \dots \times 2$ (n times) as 2^n . Explain that this type of relationship is called *exponential growth*.
- 3 Ask students to make a graph of the values in the input/output table, plotting points for $n = 0$ to $n = 10$. If students have difficulty fitting in all the points on the graph, you can help them figure out that the scale on the vertical axis needs to allow values up to 1,024, while the horizontal axis goes from 1 to 10.
- 4 For the sake of comparison, ask them to plot a graph of $y = 2x$ and $y = x^2$ on the same axes. Ask students to describe similarities and differences between the graphs.

Part 2: Additional Examples of Positive Exponential Functions

Tell students that they will now work on some similar examples. In each case, they will make an input/output table, figure out what the pattern is, express it algebraically, and draw a graph.

Example 1: Instead of cutting a sheet of paper in half, ask students to start with Student Page 2 and cut out the 9×3 grid. Ask them to cut it into thirds, cut each third into thirds, and repeat cutting.

Example 2: Tell students: In an ancient kingdom, a young woman saved the life of the king's son. The king promised her any reward she could name. She asked for a chessboard and a bag of rice, and then said to the king, "Give me 1 grain of rice for the first square of the chessboard, 2 grains for the second, and 4 grains for the third, and keep doubling the number of grains until you reach the last square of the chessboard." The king said, "Is that all you want?" and instantly agreed.

Ask students if they can identify the pattern she describes and represent it with an equation. They should eventually be able to represent the relationship as $g = 2^{n-1}$, where g is the number of grains of rice on the n^{th} square. This is almost the same as the doubling pattern they found for the first paper-cutting problem. The difference is that on the first square there is 1 grain of rice (2^0), on the second square, 2 grains, (2^1), on the third square, 4 grains (2^2), and so forth. The rate of increase is the same as for the first problem, but the table of input values starts at a different point— $n = 1$ rather than $n = 0$.

Now ask students if they can figure out how many grains of rice the young woman will receive after each square's rice is counted out. They should make a table and see if they can work out the pattern.

Square #	# of Grains for Each Square	Total # of Grains for All Squares
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
n	2^{n-1}	$2^n - 1$

If students can't find an easy way to express the pattern for the total number of grains, ask them to consider the function $y = 2^n$ (1, 2, 4, 8, 16, 32...). The total is just one less than 2^n for the n^{th} square.

Part 3: How Large is Each Piece of Paper?

- 1 Start with another copy of the 8 x 8 grid from Student Page 1. First ask students to determine the area of this piece. The area can *either* be considered as $8 \times 8 = 64$ square units, or, if students recognize that each square is a 2 cm square, $16 \times 16 = 256$ sq. cm. Then ask students to cut the paper in half and determine the area of each half, then to cut again, and again find the area of each piece. Ask students to record the results in a table and a graph. Ask students which cut will produce pieces with an area equal to 1 square unit, what the area will be after the tenth cut, and what the area will be after the n^{th} cut.

Number of Cuts	Area of One Piece (in square units)
0	64
1	32
2	16
3	
4	
...	
10	$64/2^{10} = 64/1,024 = 1/16$
...	
n	$64/2^n$

2 Ask students to make a graph of this relationship from $n = 0$ to $n = 10$. Ask students to predict what the curve will look like as n gets bigger than 10. They should see that the curve will get closer and closer to $y = 0$, without ever reaching it.

3 Ask students if they can represent the pattern algebraically. They may start by writing in terms of products of 2 (in the denominator): $64/1$, $64/2$, $64/(2 \times 2)$, $64/(2 \times 2 \times 2)$, and so forth. Remind students that this pattern can be written as $64/2^n$, or, using the fact that $1/a^n = a^{-n}$, they can write it as 64×2^{-n} . Explain that this type of relationship is called *exponential decay*.

Part 4: Similar Problems Involving Exponential Decay

1 Have students repeat the cutting problem and area calculation, using a copy of the 9×3 grid (cut from Student Page 2) and cutting each piece into thirds each time. Students may not need to actually cut any pieces if they can figure out the areas without cutting. Ask them to make a table and a graph of the areas, in relation to the number of cuts, and describe the resulting pattern algebraically. They should find that they can express this relationship as $A = 27/3^n$, or 27×3^{-n} .

2 Read the following to students: A rabbit is hopping across a field that is 100 meters wide. Every time it hops, it hops half the distance from its current position to the edge of the field.

3 Ask students to make a table showing the number of hops, the distance remaining after each hop, and the total distance traveled after each hop. (In order to do this, they may wish to make a drawing and determine how far the rabbit travels with each hop.)

Number of Hops (n)	1	2	3	4	5
Distance (d) to the Far End of the Field, in Meters	50	25	12.5		
Total Distance (t) Traveled, in Meters	50	75	87.5		

4 Ask students to make a graph of both relationships, with the distance from 0 to 100 meters on the vertical axis and the number of hops on the horizontal axis. Their graph of distance remaining should get closer and closer to zero, without ever reaching zero; the graph of total distance traveled should get closer and closer to 100, without ever reaching 100.

5 Ask students to try to represent the relationships exponentially:

- ♦ The distance remaining can be expressed as a series: $100/2$, $100/4$, $100/8$, or, more generally, $100/2^n = 100 \times 2^{-n}$.
- ♦ The total distance traveled is the difference between 100 and the distance remaining: $100 - 100/2$; $100 - 100/4$; $100 - 100/8$, ... or, more generally, $100 - 100/2^n = 100(1 - 1/2^n) = 100(1 - 2^{-n})$.

Extension

Now that students have studied several different kinds of relationships, the following abstract problem might be interesting to them. Three number sequences each start with a 1 followed by a 4. After that they are different. Ask students to do the following:

- ♦ Extend each pattern three more steps
- ♦ Explain how they know what numbers to use for each series

- ♦ Tell what type of algebraic expression it is (i.e., linear, quadratic, cubic, or exponential)
- ♦ Write an algebraic expression for each sequence
- ♦ Explain their thinking

Number in Sequence	1	2	3	4	5	6
A	1	4	7			
B	1	4	9			
C	1	4	16			

Possible solutions:

Number in Sequence	1	2	3	4	5	6
A: Linear Relationship	1	4	7	10	13	16
B: Quadratic Relationship	1	4	9	16	25	36
C: Exponential Relationship	1	4	16	64	256	1,024

Possible linear relationship: $y = 3x - 2$

Possible quadratic relationship: $y = x^2$

Possible exponential relationship: $y = 4^n$

Graphs of Linear Functions

Overview

Mathematical Focus

- ▶ Comparison of graphs of linear functions
- ▶ Effects of changes in m and b on the graph of the linear equation $y = mx + b$
- ▶ Predictions of what graphs will look like without plotting points

In this activity, students use graphing calculators to explore the graphs of linear functions. Although they studied linear functions in an earlier unit, reviewing graphs in this activity lays the foundation for future graphing activities.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Graphing calculators
- ▶ Graph paper
- ▶ Student Page 3: Linear Graphing Puzzles I
- ▶ Student Page 4: Linear Graphing Puzzles II

Notes

In this activity, students will use information they learned in a previous unit on linear relationships.

Part 1: Studying Graphs of Linear Functions

- 1** Remind students of the general form of a linear relationship: $y = mx + b$. The constant, m , is called the *slope* of the line, while the constant “ b ” is called the *y-intercept*, the place where $x = 0$ and the line crosses the y -axis.

Explain that this activity will focus on graphs of linear relationships. Ask students to roughly sketch what they think the graph of $y = 2x + 5$ might look like. Then ask them to sketch several other graphs, such as the following, on the same set of axes:

- ♦ What would the graph look like for the equation $y = 2x + 2$? $y = 2x$? $y = 2x - 5$?
- ♦ What about graphs for $y = 3x + 5$? $y = 3x$? $y = 3x - 5$?
- ♦ What about graphs for $y = -2x + 5$? $y = -3x + 5$?

- 2** Have students try some experiments, using the graphing calculator. Ask them to start with the linear functions they just sketched, and use the calculator to verify their predictions.

- 3** Ask students to try more experiments with the graphing calculator. Ask students to suppose they have an equation in the form $y = mx + b$, with m and b both positive numbers. Ask questions such as:

- ♦ What happens to the graph if you use larger values of m ?
Smaller values of m ?
- ♦ What happens if you use larger values of b ? Smaller values of b ?
- ♦ What happens if m is positive and $b = 0$? What happens if b is negative?
- ♦ What happens if m is negative? What happens if m is negative and the absolute value of m increases? If it decreases?

- 5** Ask students to sketch the same graphs as they did in Part 1. Ask them to compare these graphs with their original predictions and explain any differences they see.

Part 2: Solving Graphing Puzzles

Distribute copies of Student Pages 3 and 4: Linear Graphing Puzzles. Ask students to use their graphing calculators to create designs that look like the ones in the drawings. Tell students that they may need to adjust the limits of the graph window to make their designs look like the ones in the drawings.

Here are some hints you can use if students get stuck on a puzzle:

- ♦ Puzzle #1: What are some equations of lines that go through 0? What is the value of b for a line that goes through $0, 0$? How do you get a line to slope downward to the right? How can you get a line sloping in one direction to look like the mirror image of a line in another direction?
- ♦ Puzzle #2: How can you draw four lines all parallel to one another? How can you make them evenly spaced?
- ♦ Puzzle #3: How can you make two lines that are symmetrical and pass through the same point on the y -axis?

The equations and window limits that students should end up with for each of the puzzles are as follows:

	Puzzle #1	Puzzle #2	Puzzle #3
X min =	-10	-10	-10
X max =	10	10	10
Y min =	-10	-10	-10
Y max =	10	10	10
$y =$	x	$2x - 4$	$x - 4$
$y =$	$2x$	$2x$	$x + 4$
$y =$	$-x$	$2x + 4$	$-x - 4$
$y =$	$-2x$	$2x + 8$	$-x + 4$

Graphs of Quadratic Functions

Overview

Mathematical Focus

- ▶ Graphs of Quadratic Functions
- ▶ Effects of changes in a , b , and c on graphs of quadratic equations in the form $y = ax^2 + bx + c$
- ▶ Effects of changes in h and k on graphs of quadratic equations in the form $y = (x + h)^2 + k$
- ▶ Predictions of what graphs look like without plotting points

In this activity, students use graphing calculators to explore the graphs of quadratic functions.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Graphing calculators
- ▶ Graph paper
- ▶ Student Page 5: Quadratic Graphing Puzzles I
- ▶ Student Page 6: Quadratic Graphing Puzzles II

Activity

Part 1: Studying Graphs of Quadratic Functions

- 1** Remind students of the graphs of quadratic functions they drew by hand in Activities 1 and 2. Ask them what they remember about those graphs. Students may recall that the graphs were not straight lines—that they curved upward. The graphs in Activities 1 and 2 involved only positive integers as values for both variables. Explain that the graphs in this activity will use both positive and negative values, integers and non-integers.
- 2** Remind students of the general form of a quadratic equation: $y = ax^2 + bx + c$. Tell them that they will explore what happens when these values change, beginning with equations in the form $y = ax^2$.

Teaching Tip

The y values in quadratic functions increase rapidly as x increases. Show students how to increase the limits along the y -axis so that the graphs form nice curves, cover a fair percentage of the screen, and are easier to compare. (For example, see what happens when you change the limits to $-10 \leq x \leq 10$; $-100 \leq y \leq 100$.)

Ask students what happens to the graph when a , the coefficient of x^2 , is larger, and what happens when it is smaller. Have students plot the graphs of $y = x^2$, $y = 2x^2$, and $y = \frac{1}{2}x^2$, and describe what happens.

Ask what happens when a is negative. Have students plot the graphs of $y = x^2$, $y = -x^2$ and $y = \frac{1}{2}x^2$, and $y = -\frac{1}{2}x^2$ and describe what happens. Ask students to describe any symmetry they see in the graphs they just plotted.

Remind students that graphs of quadratic relationships are called *parabolas*. They can look like a cup holding water if they curve upward and a cup spilling water if they curve downward. The line of symmetry is called the *axis* of the parabola. The highest or lowest point on the line of symmetry is called the *vertex* of the parabola.

3 Ask students to add a constant, c , and explore quadratic relationships of the type $y = ax^2 + c$. Ask them to draw graphs and answer the following questions:

- ♦ *What happens to the line of symmetry and the vertex of a parabola when a is positive and c increases? When c decreases?*
- ♦ *What happens to the line of symmetry and the vertex of a parabola when a is negative and c increases? When c decreases?*
- ♦ *How does the shape of the curve change when c changes?*
- ♦ *What do you observe if you increase and decrease a , while keeping c constant?*

4 Ask students to consider a complete quadratic equation with all terms, $ax^2 + bx + c$. Have them begin by keeping a and c constant. Ask them the following questions:

- ♦ *What happens to the axis and vertex of a parabola when b increases? When b decreases?*
- ♦ *If a is negative, what happens to the axis and vertex of a parabola when b increases? When b decreases?*
- ♦ *How does the shape of the parabola change when b changes?*
- ♦ *What do you observe if you increase and decrease a , while keeping b and c constant?*

- 5** Ask students to consider a quadratic equation in a different form, $y = a(x + h)^2 + k$. Have them explore what happens to the shape, the axis, and the vertex as they change h , k , and a . Ask them to start by keeping a constant and changing h and k . For example: $y = (x + 5)^2 + 3$; $y = (x - 5)^2 + 3$; $y = (x + 5)^2 - 3$; $y = (x - 5)^2 - 3$. Ask students what happens to the shape, the axis and the vertex as you increase and decrease h , as you increase or decrease c , and as you increase or decrease a .
- 6** Ask students to summarize what they noticed about how changing the constants changes the graphs of parabolas.

Teaching
Tip

- The constant “ a ” determines the shape: how sharply or broadly the parabola curves, and whether it “holds water” (curves upward) or “spills water” (curves downward).
- The constant “ b ” moves the line of symmetry to the left or right; h does the same for the form $y = a(x + h)^2 + k$
- c determines where the parabola crosses the y axis; in effect, fixing the height of the parabola on the y axis; k determines the height of the vertex for the form $y = a(x + h)^2 + k$.

Part 2: Solving Quadratic Graphing Puzzles

Distribute copies of Student Pages 5 and 6: Quadratic Graphing Puzzles. Ask students to use their graphing calculators to create designs that look like the ones in the drawings. Tell students that they may need to adjust the limits of the graph window to make their designs look like the ones in the drawings.

Here are some hints you can use if students get stuck on a puzzle.

- ♦ Puzzle #1: How can you make a parabola go through $0, 0$?
How can you draw a parabola higher or lower on the y -axis?

- ♦ Puzzle #2: How can you make one parabola curve more widely than another? How can you draw two parabolas with the same shape, except that one curves downward?

- ♦ Puzzle #3: Use the same hints that you did in Puzzles 1 and 2.

- ♦ Puzzle #4: This puzzle is easier to solve if you use the form $y = a(x + h)^2 + k$. How can you draw two parabolas with different axes, evenly spaced on both sides of the y-axis? How can you draw a parabola with a vertex at $y = 0$ (on the x-axis)?

The equations and window limits that students should end up with for each of the puzzles are as follows:

	Puzzle #1	Puzzle #2	Puzzle #3	Puzzle #4
X min =	-6	-6	-6	-14
X max =	6	6	6	14
Y min =	-10	-12	-12	-10
Y max =	20	12	12	36
y =	$x^2 - 5$	$x^2 - 5$	x^2	$(x + 9)^2$
y =	x^2	$x^2 - 10$	$x^2/3$	$(x + 3)^2$
y =	$x^2 + 5$	$x^2 + 5$	$-x^2$	$(x - 3)^2$
y =	$x^2 + 10$	$x^2 + 10$	$-x^2/3$	$(x - 9)^2$

Graphs of Cubic Functions

Overview

Mathematical Focus

- ▶ Graphs of cubic functions
- ▶ Affects of changes in a , h , and c on graphs of quadratic equations in the form $y = ax^3 + c$ and $y = a(x + h)^3 + c$
- ▶ Predictions of what a graph will look like without having to plot it

In this activity, students use graphing calculators to explore the graphs of quadratic functions.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Graphing calculators
- ▶ Graph paper
- ▶ Student Page 7: Cubic Graphing Puzzles I
- ▶ Student Page 8: Cubic Graphing Puzzles II

Part 1: Studying Graphs of Cubic Functions

1 Tell Students that they are going to explore graphs of cubic functions in the form $y = ax^3 + c$. First, have them set c equal to zero and explore what happens to the shape of the graph when they increase or decrease the value of a . Ask what happens when a has negative values and what happens to the position of the graph when you increase or decrease c .

2 Tell students that they are now going to explore graphs of cubic functions in the form $y = a(x + h)^3 + k$. Have them set a equal to 1. Ask what happens to the shape of the curve as h increases or decreases and what happens to the position of the graph as k increases or decreases.

Now have students keep h and k constant. Ask what happens to the shape of the graph as a increases or decreases and what happens when a is negative.

3 Have students describe how these curves are similar to parabolas and how they are different. Similarities include: both get narrower [steeper] or wider [shallower] when a increases or decreases; both move up and down on the y -axis when c increases and decreases; both curve upward [or downward] very quickly as x increases; and both move left or right as the value of h changes. Some differences that students might observe: the curves are different shapes—the function curves both upward and downward and has a vertex at which the curve changes direction; cubic graphs do not have an axis of symmetry [technically, they have 180-degree rotational symmetry about their vertex—i.e., if you rotate the curve 180 degrees, it will appear unchanged—but students most likely will not notice this]. Functions have something in common with linear functions: (They curve upward to the right for positive values of a and downward to the left for negative values of a .)

Part 2: Solving Cubic Graphing Puzzles

Distribute copies of Student Pages 7 and 8: Cubic Graphing Puzzles. Ask students to use their graphing calculators to create designs that look like the ones in the drawings. Tell them that they may need to adjust the limits of the graph window to make their designs look like the ones in the drawings.

Here are some hints you can use if students get stuck on a puzzle:

- ♦ Puzzle #1: How can you make a function go through 0, 0?
How can you draw it so that it is wider or narrower?
How can you make two functions into mirror images?
- ♦ Puzzle #2: How can you make several functions that intersect the y-axis at different places?
- ♦ Puzzle #3: Use the same hints that you did for Puzzles 1 and 2.
- ♦ Puzzle #4: This puzzle is easier to solve if you use the form $y = a(x + h)^2$. How can you draw two functions that are mirror images of each other? How can you draw two functions so that they are evenly spaced on both sides of the y-axis?

The equations and window limits that students should end up with for each of the puzzles are as follows:

	Puzzle #1	Puzzle #2	Puzzle #3	Puzzle #4
X min =	-10	-10	-10	-10
X max =	10	10	10	10
Y min =	-100	-100	-100	-50
Y max =	100	100	100	50
y =	x^3	$x^3 + 60$	$x^3 + 50$	$(x + 4)^3$
y =	$x^3/9$	$x^3 + 30$	$x^3 - 50$	$-(x + 4)^3$
y =	$-x^3$	x^3	$-x^3 + 50$	$(x - 4)^3$
y =	$-x^3/9$	$x^3 - 30$	$-x^3 - 50$	$-(x - 4)^3$

Graphs of Inverse Functions

Overview

Mathematical Focus

- ▶ Graphs of inverse functions
- ▶ For inverse equations in the form $y = a/x$, the way changes in a affect the graph
- ▶ Predictions of what graphs will look like, without plotting them

In this activity, students use graphing calculators to explore the graphs of inverse functions.

Preparation and Materials

Before this session, gather the following materials:

- ▶ Graphing calculators
- ▶ Graph paper
- ▶ Student Page 9: Inverse Graphing Puzzles I
- ▶ Student Page 10: Inverse Graphing Puzzles II

Part 1: Studying Graphs of Inverse Functions

- 1** Remind students that they studied examples of inverse functions in Activity 5, all took the form $y = a/x$ or $xy = a$. Explain that these are called *inverse relationships* because as x gets larger, y gets smaller and vice versa; they could just as easily be written as $x = a/y$.

As students explore inverse relationships with the graphing calculator, the window settings will make a big difference in what they see. For example, ask them to do the following:

- ♦ Graph $y = 1/x$, using window limits of $0 \leq y \leq 50$ $0 \leq x \leq 50$; then change to window limits of $0 \leq y \leq 5$ $0 \leq x \leq 5$. Ask students why the two graphs look so different, what happens to values of y when x grows very large, and what happens to values of x when y grows very large.
- ♦ Graphing $y = 1/x$, using window limits of $-5 \leq y \leq 5$ $-5 \leq x \leq 5$. Ask them to describe what happens to y when x is negative. Ask students if the graph have a line of symmetry, and if so then how many lines of symmetry does it have.
- ♦ Graph the “family” of functions, $y = 1/x$, $y = 2/x$, $y = 4/x$, and $y = 8/x$, on the same axes. Students may want to increase the window limits to see a good portion of all four graphs. Ask students how the graph changes when a increases and where the lines of symmetry for each graph are.

Ask students to predict what they will see when they graph $y = -1/x$. Have them try graphing a family of functions: $y = -1/x$, $y = -2/x$, $y = -4/x$ and $y = -8/x$. Have students explain why the graph changes when a is negative and where the lines of symmetry are.

Ask students to predict what will happen when they graph $y = 2/x$, $y = -2/x$, $y = 4/x$, and $y = -4/x$. Ask them where the lines of symmetry will be.

Part 2: Solving Inverse Graphing Puzzles

Distribute copies of Student Pages 9 and 10: Inverse Graphing Puzzles. Ask students to use their graphing calculators to create designs that look like the ones in the drawings. Tell students that they may need to adjust the limits of the graph window to make their designs look like the ones in the drawings.

Here are some hints you can use if students get stuck on a puzzle:

- ♦ Puzzle #1: How can you set the window so that you only see one set of curves? How can you make the curves very close to one another? What happens if you change the window limits?
- ♦ Puzzle #2: What window limits do you need to see the upper left quadrant of the screen? What inverse functions will draw curves in that quadrant? What is similar about these two puzzles?
- ♦ Puzzle #3: How can you set the window limits to show all four quadrants of the screen? How many equations should you graph to show these curves?
- ♦ Puzzle #4: How can you adjust the window and the graphs so that the curves appear to be far from the center? How can you get two graphs that appear close to each other? How many equations should you graph to show these curves?

The equations and window limits that students should end up with for each of the puzzles are as follows:

	Puzzle #1	Puzzle #2	Puzzle #3	Puzzle #4
X min =	0	0	-6	-6
X max =	6	-6	6	6
Y min =	0	0	-6	-6
Y max =	6	6	6	6
y =	$1/x$	$-1/x$	$2/x$	$8/x$
y =	$1.5/x$	$-1.5/x$	$-2/x$	$-8/x$
y =	$2/x$	$-2/x$		$10/x$
y =	$2.5/x$	$-2.5/x$		$-10/x$

Graphs of Exponential Functions

11

Overview

Mathematical Focus

- ▶ Graphs of exponential functions
- ▶ For exponential functions in the forms $y = a^x$ and $y = a^{-x}$, understanding the way changes in a affect the graph
- ▶ Symmetries in the behavior of $y = a^x$ for values of $a < 1$ and $y = a^{-x}$ for values of $a > 1$

In this activity, students use graphing calculators to explore the graphs of exponential functions. They also compare graphs of the same function for positive and negative values of x , and find ways to construct symmetrical graphs using reciprocals.

Preparation and Materials

Before the session, gather the following materials:

- ▶ Graphing calculators
- ▶ Graph paper
- ▶ Student Page 11: Exponential Graphing Puzzles I
- ▶ Student Page 12: Exponential Graphing Puzzles II

Notes

Exponential functions of the type $y = a^x$ behave very differently from other functions students have explored in this unit. Because the functions increase and decrease so quickly, the curves can be very difficult to compare unless the ranges of x and y are chosen carefully. Therefore, a major part of this exploration concerns changing the window limits on the graphing calculator. Students do not need to *master* the behavior of exponential functions in a first-year algebra course. This activity should be thought of as an introduction to the graphs of exponential functions, which students will learn more about in later courses in mathematics, science, and social sciences.

Activity

Part 1: Exploring Graphs of Exponential Functions

- 1 Remind students that they explored exponential growth and decay in Activity 6. Ask if they can predict what the graph of $y = 2^x$ will look like. Then ask if they can predict what the curve of $y = 2^{-x}$ will look like. (These are the functions they studied when cutting up paper squares. $y = 2^x$ represented the *number* of squares after each successive cut; $y = 2^{-x}$ represented the *area* of each square after each successive cut.) Ask students to sketch their predictions on graph paper. Then have students graph both of those curves with a graphing window of $0 \leq x \leq 4$; $0 \leq y \leq 10$.

Ask students to sketch the graphs on graph paper, using the same set of axes they used for their predictions. Ask students if the graph of $y = 2x$ increased faster or slower than they thought it would and if the graph of $y = 2^{-x}$ decreased faster or slower than they thought it would.

- 2 Tell students that they will now explore what happens to the relationship $y = a^x$ for different values of a . Have them start by setting the graph window to $0 \leq x \leq 5$; $0 \leq y \leq 20$. Next, ask them to sketch, on graph paper, what they think the following graphs might look like: $y = 1^x$, $y = 1.5^x$, $y = 2^x$, and $y = 4^x$. Then have students use the graphing calculator to graph the functions. Ask them to compare their sketches with what the graphing calculator displayed.

Teaching Tip

If students are surprised that the graph of $y = 1^x$ is a straight, horizontal line, ask them to calculate 1^2 , 1^3 , 1^4 , etc. The result is $y = 1$, for any value of x .

Ask students to change the limits and see what the graph looks like if they double the upper limit for x , then y , then both. Then have them halve the original upper limit for x , then y , then both.

Tell students that they will now explore what happens for values of $a < 1$. Have them set the graph window to $0 \leq x \leq 5$; $0 \leq y \leq 2$. Ask them to graph the functions $y = 1^x$, $y = (2/3)^x$, $y = (1/2)^x$, and $y = (1/4)^x$.

Teaching
Tip

Students will have to use parentheses to enter these fractional values of a on their calculators. Fractions have been chosen rather than decimals because these values are the reciprocals of the values of a used in the previous exploration.

Ask students to sketch what they think these graphs might look like, on graph paper. Once they have used the graphing calculator, ask them to compare their sketches with what the graphing calculator displayed.

3 Tell students that they will now explore graphs of $y = a^x$ for positive and negative values of x .

a. Have them start by graphing the same functions as in the previous exploration, — $y = 1^x$, $y = (2/3)^x$, $y = (1/2)^x$, and $y = (1/4)^x$,— but this time, they should change the limits so that x has both positive and negative values. Have them set the graph window to $-5 \leq x \leq 5$; $0 \leq y \leq 2$.

Ask students to sketch the results on graph paper. Ask students why they think that y increases when x is negative. You may have to remind them that a^{-x} is the same as $1/a^x$ or $(1/a)^x$. In other words, raising any number to a negative power is the same as raising its reciprocal to a positive power. This means that $(1/4)^{-2}$ (for example) is the same as 4^2 , or 16; similarly, $(1/4)^{-3}$ is the same as 4^3 , or 64. As the magnitude of the negative power increases, the value of $y = a^x$ increases when $a < 1$ and x is negative.

Tell students to explore what happens when they change the window limits for x and y . Remind them that when they change the limits for x , they must be sure to keep the limits symmetrical about the y -axis.

b. Explain that the next step is to compare this result with what happens for values of $a > 0$, and positive and negative values of x . Tell them to keep the window limits at $-5 \leq x \leq 5$, $0 \leq y \leq 2$, and to graph the first set of functions: $y = 1^x$, $y = 1.5^x$, $y = 2^x$, and $y = 4^x$. The results should be a mirror image of what they got last time (in 3a).

Have students explore what happens when they change the window limits for x and y . Remind them that when they change the limits for x , they must be sure to keep the limits symmetrical about the y -axis.

Part 2: Solving Exponential Graphing Puzzles

Distribute copies of Student Pages 11 and 12: Exponential Graphing Puzzles. Ask students to use their graphing calculators to create designs that look like the ones in the puzzles. Tell students that they may need to adjust the limits of the graph window to make their designs look like the ones in the drawings. Also, remind them that all of the designs are made by using some of the seven functions they explored earlier.

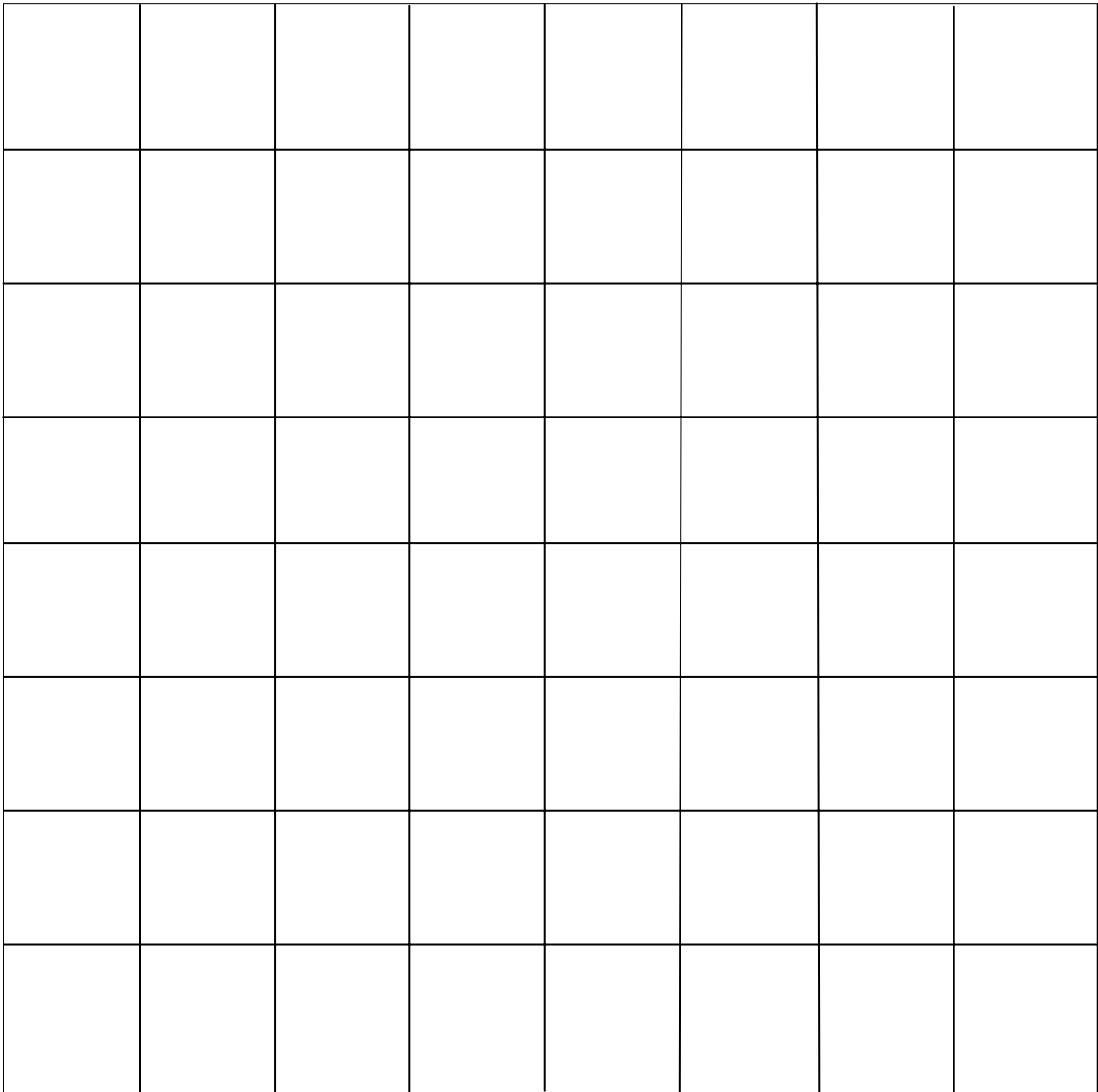
Here are some hints you can use if students get stuck on a puzzle:

- ♦ Puzzle #1: How can you make two exponential functions be mirror images of each other? Are the x limits wide or narrow for this puzzle? How can you tell what the y limit is?
- ♦ Puzzle #2: How can you tell what the x and y limits might be for this puzzle?
- ♦ Puzzle #3: What limits do you have to use to make a graph that goes right to the upper corners of the screen? What is 2^x for $x = 1, 2, 3$, and 4 ?

The equations and window limits that students should end up with for each of the puzzles are as follows:

	Puzzle #1	Puzzle #2	Puzzle #3
X min =	-2	-3	-3
X max =	2	3	3
Y min =	0	0	0
Y max =	4	2	8
y =	1^x	$(1/2)^x$	$(1/4)^x$
y =	2^x	$(2/3)^x$	$(1/2)^x$
y =	$(1/2)^x$	1.5^x	2^x
y =		2^x	4^x

8 x 8 Grid

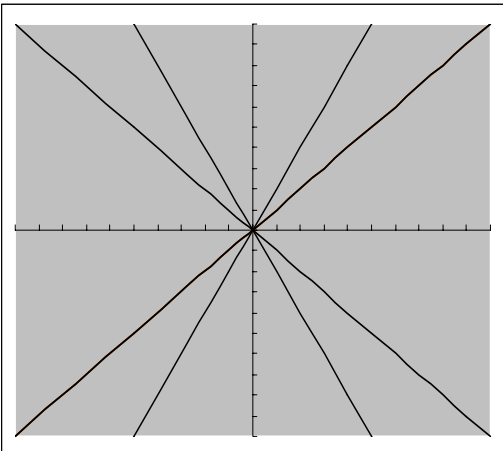


9 x 3 Grid

Linear Graphing Puzzles I

Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.

Puzzle #1: Starburst



1. The window limits are:

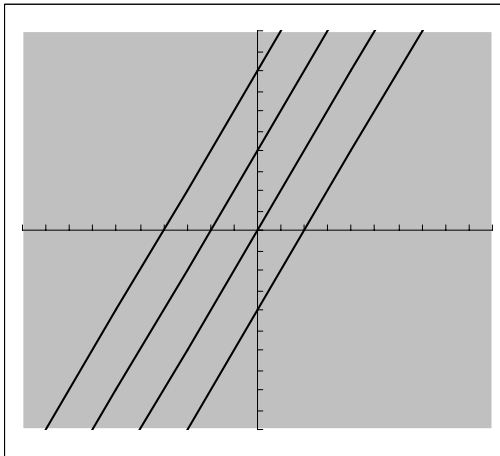
$$\underline{\quad} \leq x \leq \underline{\quad}; \quad \underline{\quad} \leq y \leq \underline{\quad}$$

2. The equations I used are:

a. $y =$	b. $y =$
c. $y =$	d. $y =$

3. I used the following relationships to solve this puzzle:

Puzzle #2: Rainstorm



1. The window limits are:

$$\underline{\quad} \leq x \leq \underline{\quad}; \quad \underline{\quad} \leq y \leq \underline{\quad}$$

2. The equations I used are:

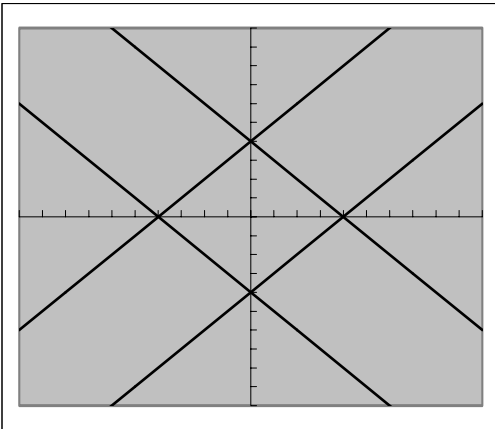
a. $y =$	b. $y =$
c. $y =$	d. $y =$

3. I used the following relationships to solve this puzzle:

Linear Graphing Puzzle II

Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.

Puzzle #3: Diamonds



1. The window limits are:

$$\underline{\quad} \leq x \leq \underline{\quad}; \quad \underline{\quad} \leq y \leq \underline{\quad}$$

2. The equations I used are:

a. $y =$

b. $y =$

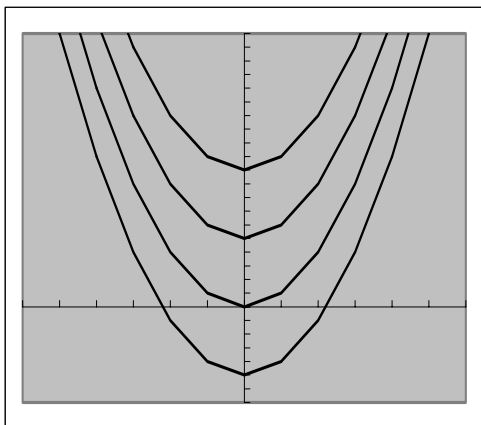
c. $y =$

d. $y =$

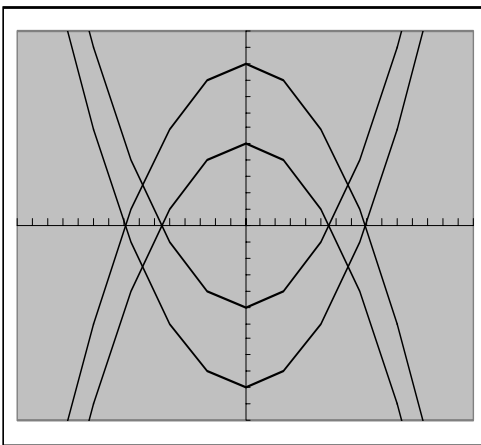
3. I used the following relationships to solve this puzzle:

Quadratic Graphing Puzzles I

Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



1. The window limits are:
 $\underline{\quad} \leq x \leq \underline{\quad}$; $\underline{\quad} \leq y \leq \underline{\quad}$
2. The equations I used are:
 a. $y =$ b. $y =$
 c. $y =$ d. $y =$
3. I used the following relationships to solve this puzzle:

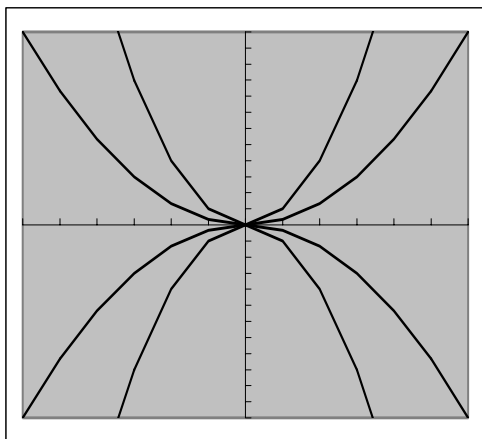


1. The window limits are:
 $\underline{\quad} \leq x \leq \underline{\quad}$; $\underline{\quad} \leq y \leq \underline{\quad}$
2. The equations I used are:
 a. $y =$ b. $y =$
 c. $y =$ d. $y =$
3. I used the following relationships to solve this puzzle:

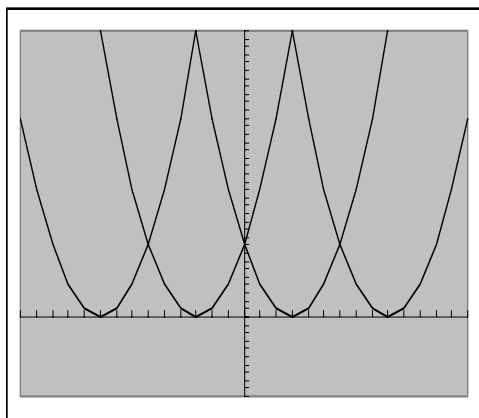
Quadratic Graphing Puzzles II



Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



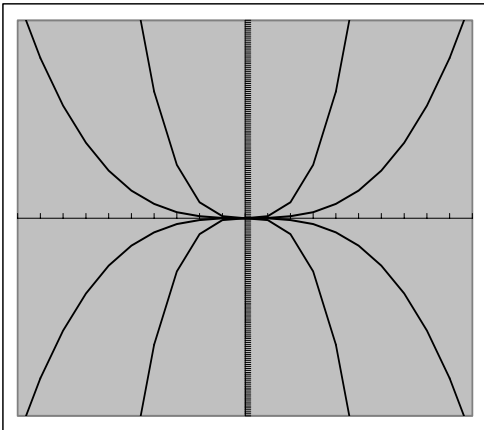
- The window limits are:
 $\underline{\quad} \leq x \leq \underline{\quad}$; $\underline{\quad} \leq y \leq \underline{\quad}$
- The equations I used are:
 a. $y =$ b. $y =$
 c. $y =$ d. $y =$
- I used the following relationships to solve this puzzle:



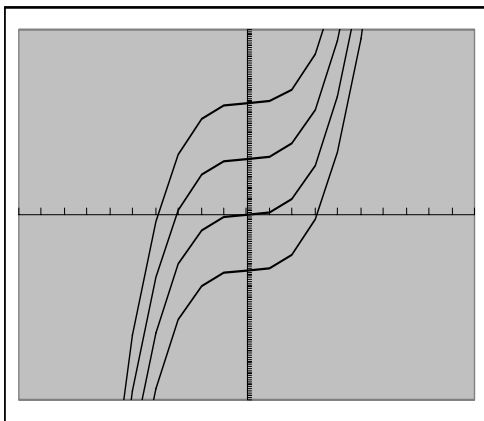
- The window limits are:
 $\underline{\quad} \leq x \leq \underline{\quad}$; $\underline{\quad} \leq y \leq \underline{\quad}$
- The equations I used are:
 a. $y =$ b. $y =$
 c. $y =$ d. $y =$
- I used the following relationships to solve this puzzle:

Cubic Graphing Puzzles I

Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



1. The window limits are:
 $___ \leq x \leq ___;$ $___ \leq y \leq ___$
2. The equations I used are:
 a. $y =$ b. $y =$
 c. $y =$ d. $y =$
3. I used the following relationships to solve this puzzle:

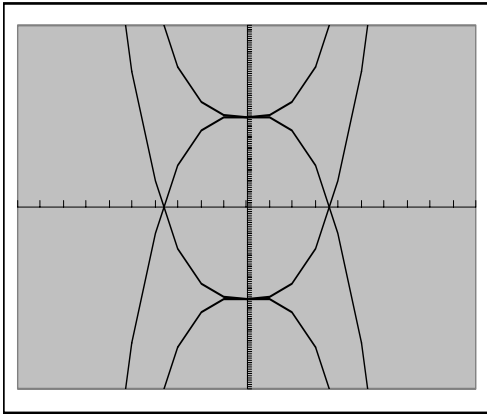


1. The window limits are:
 $___ \leq x \leq ___;$ $___ \leq y \leq ___$
2. The equations I used are:
 a. $y =$ b. $y =$
 c. $y =$ d. $y =$
3. I used the following relationships to solve this puzzle:

Cubic Graphing Puzzles II

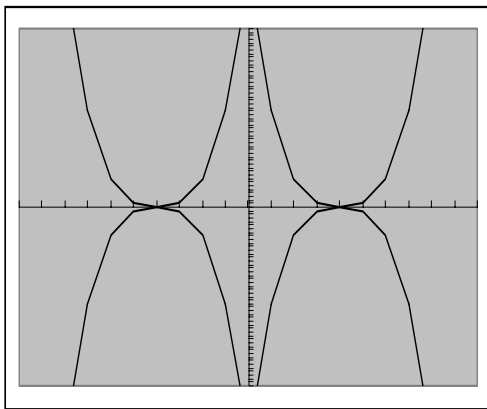


Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



1. The window limits are:
 $___ \leq x \leq ___;$ $___ \leq y \leq ___$
2. The equations I used are:

a. $y =$	b. $y =$
c. $y =$	d. $y =$
3. I used the following relationships to solve this puzzle:

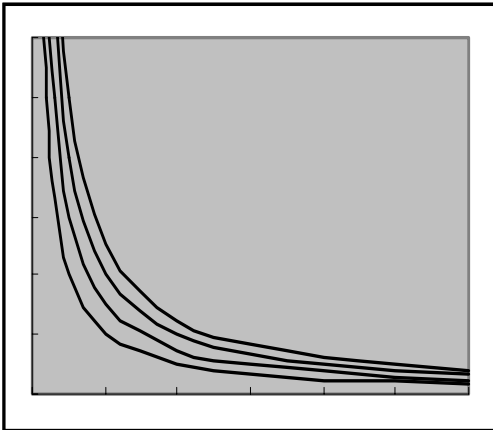


1. The window limits are:
 $___ \leq x \leq ___;$ $___ \leq y \leq ___$
2. The equations I used are:

a. $y =$	b. $y =$
c. $y =$	d. $y =$
3. I used the following relationships to solve this puzzle:

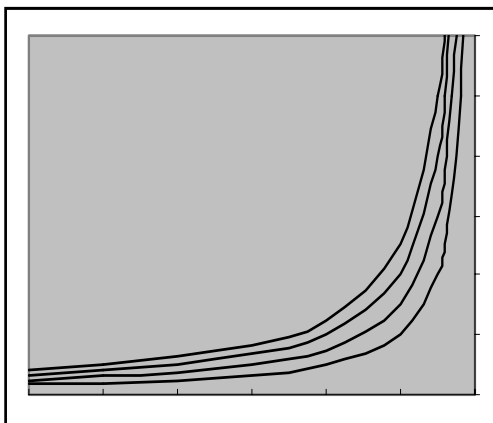
Inverse Graphing Puzzles I

Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



- The window limits are:
 $___ \leq x \leq ___ ; ___ \leq y \leq ___$
- The equations I used are:

a. $y =$	b. $y =$
c. $y =$	d. $y =$
- I used the following relationships to solve this puzzle:



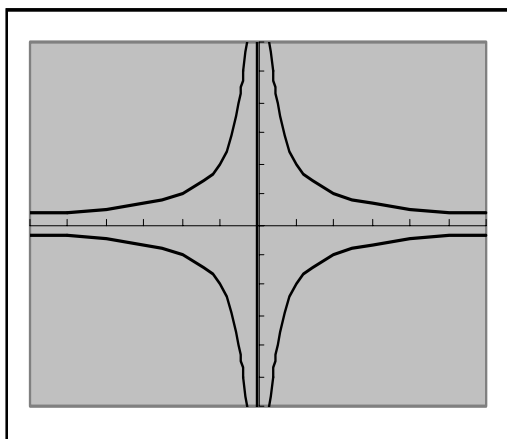
- The window limits are:
 $___ \leq x \leq ___ ; ___ \leq y \leq ___$
- The equations I used are:

a. $y =$	b. $y =$
c. $y =$	d. $y =$
- I used the following relationships to solve this puzzle:

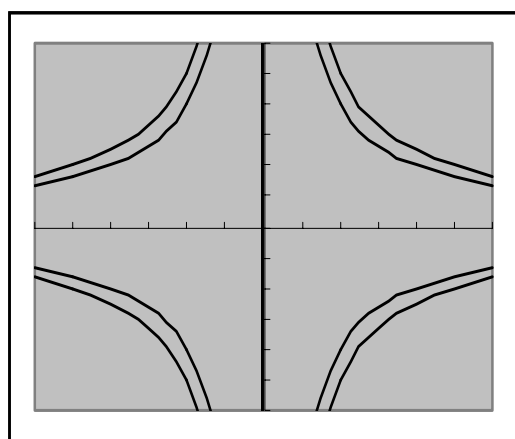
Inverse Graphing Puzzles II

11

Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



1. The window limits are:
 $___ \leq x \leq ___;$ $___ \leq y \leq ___$
2. The equations I used are:
a. $y =$ b. $y =$
3. I used the following relationships to solve this puzzle:

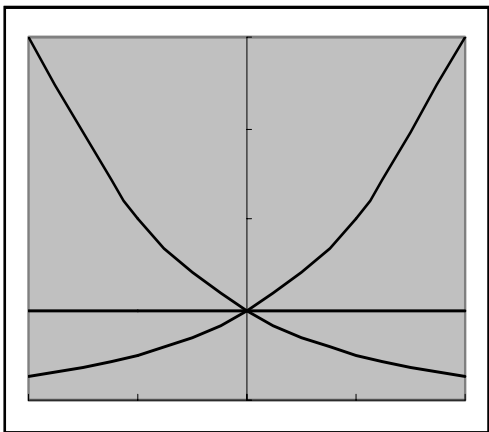


1. The window limits are:
 $___ \leq x \leq ___;$ $___ \leq y \leq ___$
2. The equations I used are:
a. $y =$ b. $y =$
c. $y =$ d. $y =$
3. I used the following relationships to solve this puzzle:

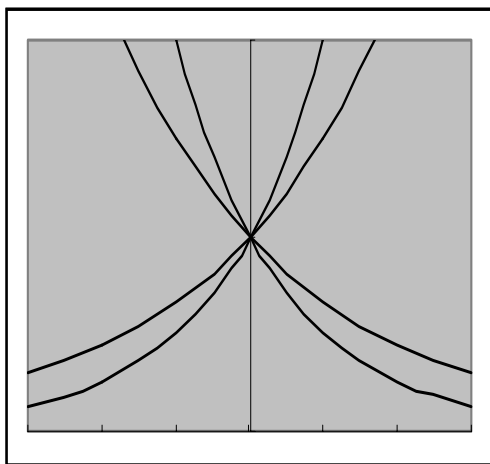
Exponential Graphing Puzzles I



Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



- The window limits are:
 $\underline{\quad} \leq x \leq \underline{\quad}$; $\underline{\quad} \leq y \leq \underline{\quad}$
- The equations I used are:
 a. $y =$ $\quad\quad\quad$ b. $y =$ $\quad\quad\quad$
 c. $y =$ $\quad\quad\quad$
- I used the following relationships to solve this puzzle:

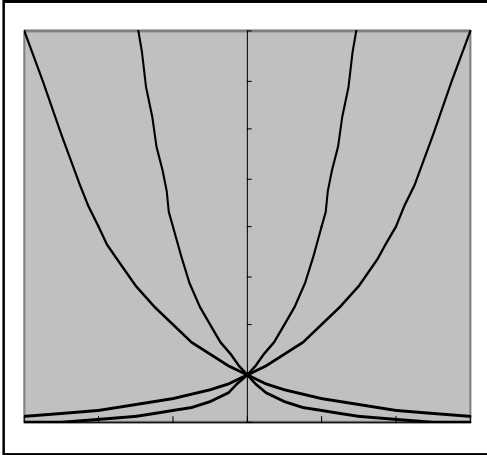


- The window limits are:
 $\underline{\quad} \leq x \leq \underline{\quad}$; $\underline{\quad} \leq y \leq \underline{\quad}$
- The equations I used are:
 a. $y =$ $\quad\quad\quad$ b. $y =$ $\quad\quad\quad$
 c. $y =$ $\quad\quad\quad$ d. $y =$ $\quad\quad\quad$
- I used the following relationships to solve this puzzle:

12

Exponential Graphing Puzzles II

Use a graphing calculator to construct designs like the puzzles on this page. After you solve each puzzle, complete the box at the right.



4. The window limits are:

$$\underline{\quad} \leq x \leq \underline{\quad}; \quad \underline{\quad} \leq y \leq \underline{\quad}$$

5. The equations I used are:

a. $y =$ b. $y =$

c. $y =$ d. $y =$

6. I used the following relationships to solve this puzzle: