Closure of “Root Three Numbers” Problem

1 Conjectures about closure: integers

If all whole numbers are arranged in three columns this way:

Row	**A**	**B**	**C**
0	0	1	2
1	3	4	5
2	6	7	8
3	9	10	11
4	12	13	14
5	15	16	17
6	18	19	20
7	21	22	23
8	24	25	26
9	27	28	29
⋮	⋮	⋮	⋮

1. In which column would you find 43? 51? 98? 300?

2. A set is called **closed under multiplication** if the product of any two numbers in that set is also in that set.
  (a) Is the set in Column **A** closed under multiplication?
  (b) What about the other two columns?

For now, just experiment by choosing pairs of numbers from one column and seeing if their product is also in that column.

3. Which of the sets, if any, seem **closed under addition**?

A set is “closed under addition” if the sum of any two numbers in that set is also in that set.

4. Which of these sets—**A**, **B**, or **C**—if any, appear to be closed under *both* addition and multiplication?

2 A proof of closure, from algebra

By looking at *specific* numbers in each column, like 9 and 15, some sets *appeared to be* closed under addition or multiplication. But, will that observation hold for *all* numbers in those sets? To show that things really *are* as they appear to be—and to show *why* they are—you need a way to go beyond *specific* numbers. You need a way to talk about numbers in a more general way.

The few numbers that we *see* in Column **A** are all multiples of 3, but the way the table is laid out shows why all the rest *must* be, too. So, $3m$ and $3n$ are two *generic* numbers in Column **A** (as long as we agree that m and n are whole numbers).

1. Using this scheme, how might you describe each number in Column **B**?

2. In a similar way, come up with a way to describe each number in Column **C**.

There are at least two sensible but different ways to answer this problem, and people choose one or the other mostly as a matter of taste. Do you see both?

3. Use the algebraic descriptions of numbers in the columns to show that the sum of any two numbers from Column **C** will always be a number in Column **B**.

Hints contain a proof that the sum of any two odd numbers is even. If you like, you can modify that proof to solve this problem.

4. Use the same method to prove any conjectures you made about Column **B** in problems 2 and 4 from section 1.

3 A new set: "Root Three Numbers"

Consider a new set of numbers: $(a+b\sqrt{3})$, where a and b are integers (numbers in the set $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$). This new set has no official name, so, to talk about it, we'll need to invent a name. Let's call it "The Root Three Numbers."

1. Which of these are Root Three Numbers?

- (a) $4-6\sqrt{3}$
- (b) $2+3\sqrt{6}$
- (c) $2+(2\sqrt{2})\sqrt{6}$
- (d) $-\sqrt{3}$
- (e) 5
- (f) $3\frac{1}{3}$
- (g) $2.5+4\sqrt{3}$
- (h) $3\sqrt{3}-5$
- (i) 0
- (j) $1+\frac{\sqrt{3}}{2}$

2. Which of these products produces a Root Three Number?

- (a) $(1+2\sqrt{3})(3-4\sqrt{3})$
- (b) $(2-\sqrt{3})^2$
- (c) $(5+3\sqrt{3})(5-3\sqrt{3})$
- (d) $(-2+\sqrt{3})(-2-\sqrt{3})$
- (e) $(10+q\sqrt{3})(10-q\sqrt{3})$
- (f) $(3+2\sqrt{3})(6-4\sqrt{3})$
- (g) $(p+2\sqrt{3})(-p+2\sqrt{3})$
- (h) $(7+4\sqrt{3})(p-q\sqrt{3})$
- (i) $(p+q\sqrt{3})(r+s\sqrt{3})$
- (j) $(p+q\sqrt{3})(p-q\sqrt{3})$

In these problems, p, q, r, and s are always integers.

3. On the basis of your results in problem 2, does this set of numbers seem to be closed under multiplication?

Which example in problem 2 *proves* that the set is or is not closed?

4. Using one "generic" example, show that the Root Three Numbers are closed under addition.

5. Some products in problem 2 contained a $\sqrt{3}$ and some were ordinary integers. Find three more pairs of Root Three Numbers whose products are ordinary integers. (Don't use 0 for a or b.) What rule are you using to create your pairs?

In problem 5, you looked at pairs of *non*-integers whose sums and products *are* integers. Numbers like this—ones that are *not* part of a set (in this case, the integers), but whose sum and product both *are* part of that set—are so important in mathematics that they have a name: such numbers are called **conjugates** of each other.

You will meet conjugates again when you study complex numbers.

6. In problems 3–5 you found three properties of the set of Root Three Numbers: two closure properties, and the existence of conjugate pairs. Do "Root Two Numbers" (numbers of the form $a+b\sqrt{2}$) have the same properties? *Prove* your claims.

What about a set consisting of numbers of the form $a+b\sqrt{-1}$?

Hints

1 Conjectures about closure: integers

There are no hints for this section.

2 A proof of closure, from algebra

Hint to problem 1. Numbers in Column **B** are 1 greater than numbers in Column **A**.

Hint to problem 2. Numbers in Column **C** are 2 greater than those in Column **A**. Also, they are 1 less than numbers in Column **A**.

Hint to problem 3. This is how you might prove that the sum of two odd numbers is always even. Let $2m+1$ and $2n+1$ be the two odd numbers (with m and n chosen to be integers). Their sum is $2m+2n+2$. That can be factored as $2(m+n+1)$, which is even, because it is 2 times some integer.

You can apply this *style* of proof to your problem.

Hint to problem 4. After finding the product, factor part of it in a new way.

3 A new set: "Root Three Numbers"

Hint to problem 3. Look at example (i), the generic case.

Answers

1 Conjectures about closure: integers

1. Column **A** contains all of the multiples of 3, including 51 and 300. Column **B** contains 43. Column **C** contains 98.

2. (a) The set in Column **A** appears closed under multiplication.

(b) Column **B** also seems closed under multiplication. But Column **C** is not closed under multiplication.

3. It looks like Column **A** is closed under addition. Neither of Columns **B** or **C** is closed under addition.

4. Only the set in Column **A** appears to be closed under *both* addition and multiplication.

2 A proof of closure, from algebra

1. A good algebraic (generic) description of numbers in Column **B** is $3r + 1$.

2. Numbers in Column **C** might be described as $3n + 2$ or $3n - 1$.

3. See solution.

4. See solution.

3 A new set: “Root Three Numbers”

1. The following are *not* Root Three Numbers. The rest are.

(b) $2 + 3\sqrt{6}$

(f) $3\frac{1}{3}$

(g) $2.5 + 4\sqrt{3}$

(j) $1 + \frac{\sqrt{3}}{2}$

2. All of the products are Root Three Numbers.

3. Yes, this set is closed under multiplication. See solution for explanation.

4. See solution.

5. See solution.

6. The Root Two Numbers are also closed under addition and multiplication, and have conjugate pairs (pairs of non-integer Root Two Numbers whose sums and products are ordinary integers). See solution for proof.

Solutions

1 Conjectures about closure: integers

1. To find 43, 51, 98, and 300, you could just extend the table and look, but that is a nuisance and would be impractical if the problem had asked where 3,000,000,001 can be found. Instead, notice that Column **A**'s numbers are all multiples of 3, and that Columns **B** and **C** contain the next two numbers in sequence. That pattern *must* continue because after any one multiple of 3 occurs in Column **A** (and we know that at least one does!), there are exactly two numbers before the next multiple of 3 occurs. Columns **B** and **C** take up those two numbers, guaranteeing that the next multiple of 3 will again be in Column **A**.

This is an argument by mathematical induction, stated informally.

So, 51 and 300 must be in Column **A** because they are both multiples of 3. Because 43 comes right after 42 (a multiple of 3), it must be in Column **B**. And 98 comes just before 99 (or two after 96), so it is in Column **C**.

2. (a) By experimentation alone (not proof) you can see that $0 \times 15 = 0$, $3 \times 6 = 18$, and so on. The set in Column **A** appears closed under multiplication.

(b) Likewise, a few experiments (like $4 \times 7 = 28$) make it seem that Column **B** is also closed under multiplication, but we don't (yet!) have proof. The product of 2 and 5 (both from Column **C**) is 10, which is *not* in Column **C**. This is enough to *prove* that Column **C** is not closed under multiplication.

3. From experiments like $3 + 24 = 27$ and $6 + 9 = 15$, it looks like Column **A** might well be closed under addition. One experiment in each of the other columns tells with certainty that Columns **B** and **C** are not closed under addition. For example, the sum of 1 and 4 from Column **B** is 5, which is not in Column **B**.

4. Only the set in Column **A** appears to be closed under *both* addition and multiplication.

2 A proof of closure, from algebra

1. If numbers in Column **A** are described generically as $3r$, then numbers in Column **B** can be described as $3r + 1$. Substituting 0, 1, 2, 3, and so on, for r gives all the numbers in Column **B**.

2. If a particular number in Column **A** happens to be $3n$, then the number in that same row, but in Column **C** will be $3n + 2$. Again, substituting whole-number values for n gives only (and all) numbers in Column **C**.

Another way to think about Column **C** is that it is always 1 less than a multiple of 3, so $3n - 1$ will also work. Again, substituting any whole number (other than 0) gives numbers in Column **C**.

3. Let's use $3m+2$ and $3n+2$ to represent two numbers from Column **C**. Their sum is

Hints show a proof that the sum of any two odds is even.

$$(3n+2)+(3m+2) = 3n+3m+4 = 3n+3m+3+1 = 3(n+m+1)+1$$

The $3(n + m + 1) + 1$ is one greater than a multiple of 3. But those are precisely the numbers in Column **B**, *quod erat demonstrandum*.

The Latin *quod erat demonstrandum*, often abbreviated Q.E.D., means "which is what was to be shown."

4. The conjecture was that Column **B** was closed under multiplication. Let two numbers from Column **B** be $3m + 1$ and $3n + 1$. Their product is

$$(3m+1)(3n+1) = 9mn+3m+3n+1 = 3(mn+m+n)+1$$

This is also one greater than a multiple of 3, again a number in Column **B**, which is what we wanted to show.

3 A new set: "Root Three Numbers"

1. Several of these do not *look like* the model $a + b\sqrt{3}$, but only (b), (f), (g), and (j) actually are not.

Expression (c) works because

$$2 + (2\sqrt{2})\sqrt{6} = 2 + (2\sqrt{2})\sqrt{2}\sqrt{3} = 2 + 4\sqrt{3}$$

which fits our definition of a Root Three Number. If a or b are 0, then the form $a + b\sqrt{3}$ generates numbers like (d), (e), and (i). For example, (i) is really $0 + 0\sqrt{3}$ and (d) is really $0 +^{-} 1\sqrt{3}$. (h) just switches the order of the two parts: $3\sqrt{3} - 5 = -5 + 3\sqrt{3}$, so it's a Root Three Number.

But (f), (g), and (j) use non-integer values for a or b, which is not allowed. The expression in (b) is harder to judge. The $\sqrt{6}$ can't be transformed to a $\sqrt{3}$ without making another part of the expression irrational (and not $\sqrt{3}$), but that is not obvious.

See the solutions of the problem sequence "Approximating irrationals with rationals" for one approach to proving that $2 + 3\sqrt{6}$ is not a Root Three Number.

2. All of the products are Root Three Numbers.

(a) $(1 + 2\sqrt{3})(3 - 4\sqrt{3}) = -21 + 2\sqrt{3}$
(b) $(2 - \sqrt{3})^2 = 7 - 4\sqrt{3}$
(c) $(5 + 3\sqrt{3})(5 - 3\sqrt{3}) = -2$
(d) $(-2 + \sqrt{3})(-2 - \sqrt{3}) = 1$
(e) $(10 + q\sqrt{3})(10 - q\sqrt{3}) = 100 - 3q^2$
(f) $(3 + 2\sqrt{3})(6 - 4\sqrt{3}) = -6$
(g) $(p + 2\sqrt{3})(-p + 2\sqrt{3}) = 12 - p^2$
(h) $(7 + 4\sqrt{3})(p - q\sqrt{3}) = (7p - 12q) + (4p - 7q)\sqrt{3}$
(i) $(p + q\sqrt{3})(r + s\sqrt{3}) = (pr + 3qs) + (ps + qr)\sqrt{3}$
(j) $(p + q\sqrt{3})(p - q\sqrt{3}) = p^2 - 3q^2$

3. The Root Three Numbers not only *seem* to be closed under multiplication, but example (i) is so generic that it *proves* that this set is closed under multiplication. For any integer values of p, q, s, and r, $(p + q\sqrt{3})$, $(r + s\sqrt{3})$, and $(pr + 3qs) + (ps + qr)\sqrt{3}$ will all be Root Three Numbers.

4. $(p+q\sqrt{3})+(r+s\sqrt{3}) = (p+r)+(q+s)\sqrt{2}$, which is a Root Three Number, proving that the Root Three Numbers are closed under addition.

5. Examples (c), (d), (e), (g), and (j) suggest a pattern which (j) shows in the most generic way. Pick any two integers a and b, and create the pair $(a + b\sqrt{3})$ and $(a - b\sqrt{3})$. Their product is a^2+3b^2, which must be an integer. Clearly their sum, $2a$, is also an ordinary integer.

At first, example (f) looks as if it follows a different rule, but $(3 + 2\sqrt{3})(6 - 4\sqrt{3}) = (3 + 2\sqrt{3}) \cdot 2 \cdot (3 - 2\sqrt{3})$, which follows the pattern shown above.

6. Exactly the same methods prove that "Root Two Numbers" (numbers of the form $a + b\sqrt{2}$) are also closed under addition and multiplication, and have conjugate pairs (pairs of non-integer Root Two Numbers whose sums and products *are* ordinary integers). Starting with the generic case, $(p+q\sqrt{2})(r+s\sqrt{2}) = (pr+2qs)+(ps+qr)\sqrt{2}$, we see that if $ps + qr = 0$—that is, if $ps = -qr$ or $\frac{p}{q} = -\frac{r}{s}$—then $(pr + 2qs) + (ps + qr)\sqrt{2}$ will be an integer.