EXAMPLE

Did you ever notice the pattern in the digits of $\frac{k}{7}$ $(1 \leq k \leq 6)$?

$$\frac{1}{7} = .142857142857\ldots$$

$$\frac{2}{7} = .285714285714\ldots$$

$$\frac{3}{7} = .428571428571\ldots$$

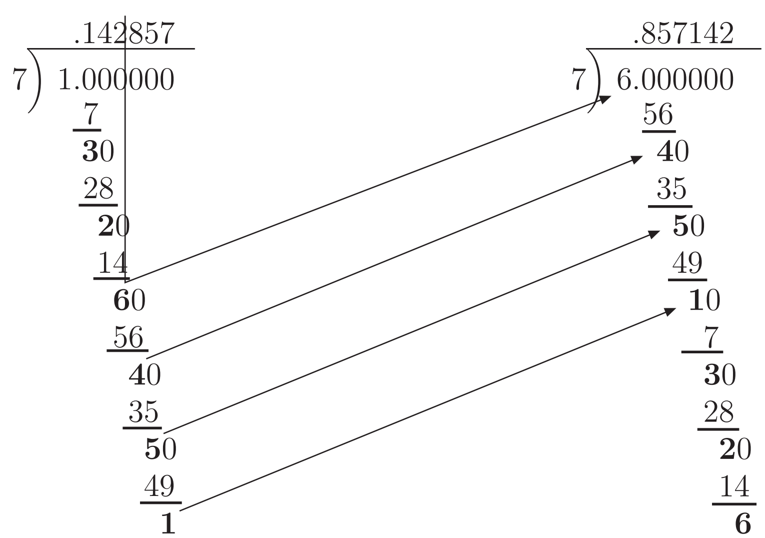
$$\frac{4}{7} = .571428571428\ldots$$

$$\frac{5}{7} = .714285714285\ldots$$

$$\frac{6}{7} = .857142857142\ldots$$

$$\begin{array}{r} .142857 \\ 7\overline{)\,1.000000} \\ \underline{7} \\ \mathbf{3}0 \\ \underline{28} \\ \mathbf{2}0 \\ \underline{14} \\ \mathbf{6}0 \\ \underline{56} \\ \mathbf{4}0 \\ \underline{35} \\ \mathbf{5}0 \\ \underline{49} \\ \mathbf{1} \end{array}$$

Notice that each of the remainders 1–6 shows up exactly once in this calculation, in this sequence: $3, 2, 6, 4, 5, 1$. Once you get a remainder of 1, the process will start over again, and the digits in the quotient $(1, 4, 2, 8, 5, 7)$ will repeat. Look down the remainder list until you get a 6 and observe an interesting connection with the decimal expansion for $\frac{6}{7}$.



1. Without doing any calculations, explain how the previous calculation of $\frac{1}{7}$ can be used to get the following decimal expansions:

(a) $\frac{2}{7}$ (b) $\frac{3}{7}$

(c) $\frac{4}{7}$ (d) $\frac{5}{7}$

(e) $\frac{12}{7}$ (f) $\frac{50}{7}$

2. Write the digits of the repetend for $\frac{1}{7}$ in a circle.
Explain how the circle of digits can help you find the decimal equivalents for $\frac{2}{7}, \frac{3}{7}, \ldots \frac{6}{7}$.

"Repetend" describes the repeating part of the decimal expansion.

There are 6 digits in the repetend for $\frac{1}{7}$ and all of the fractions $\frac{k}{7}$ ($1 \leq k \leq 6$) use the same 6 digits in their repetends. We say that the fraction $\frac{1}{7}$ has <u>1 "ring" of size 6</u>. We also say that the decimal expansion has *period* 6.

The fraction $\frac{1}{11}$ has 5 rings of size 2 (and period 2)

3. Calculate decimal expansions for each fraction using long division. For each one, what other fractions-to-decimal expansions (if any) do you get for free?

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{9}$

Oh, this looks like a lot of work, but do it carefully and savor every step. Look for shortcuts, conjectures, and theorems. Get friendly with these numbers.

(d) $\frac{1}{15}$ (e) $\frac{4}{15}$ (f) $\frac{7}{15}$

(g) $\frac{1}{8}$ (h) $\frac{1}{13}$ (i) $\frac{1}{20}$

(j) $\frac{1}{19}$ (k) $\frac{1}{21}$ (l) $\frac{1}{35}$

You're probably noticed some patterns in the decimal expansions of various fractions. Let's address some of them by considering thirteenths.

4. How many separate long divisions do you need to do before you know all the decimal expansions for $\frac{k}{13}$, $1 \leq k \leq 12$?

5. Explain how to use the two rings to find the decimal equivalents for all fractions less than 1 with denominator 13.

6. Use the fraction ring method to find the decimal equivalents for all fractions less than 1 with denominator 19. Do the same for denominators 31 and 37.

7. Make a table showing the period for the decimal expansion of $\frac{1}{n}$ for every n between 2 and 25.

8. Consider some of the following questions and make some conjectures as to their answers. How might you go about proving your conjecture?

(a) For which n will the decmial expansion of $\frac{1}{n}$ terminate?

(b) Given a positive integer n, what's the biggest the period of $\frac{1}{n}$ be?

(c) Under what circumstances will the period of $\frac{1}{n}$ equal $n-1$?

(d) Let p be a prime number. What is the relationship among the number of digits in a ring, the number of rings, and the denominator p for any fraction $\frac{m}{p}$ where $1 \leq m < p$?

(e) How do you know that a decimal expansion of every fraction either terminates or repeats?

(f) What other patterns have you noticed?

What is the period of $\frac{1}{n}$?

9. The period of $\frac{1}{7}$ is 6. What is the remainder you get when you divide 10^6 by 7? Will a smaller power of 10 have the same remainder?

10. The period of $\frac{1}{11}$ is 2. What is the remainder you get when you divide 10^2 by 2? Will a smaller power of 10 have the same remainder?

11. **Make (and test) a conjecture:**
What's the postive smallest integer power, k, for which 10^k has remainder 1 when divided by 13?

12. **What is the period of $\frac{1}{n}$?**
Let n be a positive integer divisible by neither 2 nor 5. Make and prove a conjecture for determining the period of $\frac{1}{n}$ without computing its decimal expansion.

13. **Why not divisible by 2 or 5?**
Why can't n be divisible by 2 or 5? What happens then?