Education Development Center, 2004

Problem Solving and Problem Posing

The Consecutive Sums Problem

Current mathematics curricula ask students, from time to time, to "investigate." But how does one do that?

These materials are adapted from Ways to Think About Mathematics: Activities and Investigations for Grade 6–12 Teachers*, published by Corwin Press.*

Another question: why all the fuss about investigation? Investigation is not the only way to learn mathematics, nor even the best way in *every* situation.

The ability to investigate a situation is, in itself, an important skill for students to acquire. In mathematics—as in science, or diagnosing the ills of an automobile, a computer, or a person—proper investigation is often the first step in successful problem solving. Furthermore, investigation helps to bring to the fore an essential feature of the subject itself.

Mathematics is a specialization of many of the most powerful thinking techniques people normally use. Part of its great power derives from the facts, formulas, and techniques it provides to the sciences. What makes it of value even to those who will someday forget the facts and formulas is that it highlights, extends, and refines the kinds of thinking that people do in *all* fields. These include investigation, pattern-seeking, and proof.

Proof outside of mathematics is different, in ways, from proof within the discipline, but the fact that the same word is used attests to the relatedness of the many purposes of proof, and even to similarities in the ways of thinking.

Skilled investigators in any field have strategies that go beyond poking around and hoping for the best. In investigation, as in other aspects of thinking, mathematics adds its own special features. What makes an investigation *mathematical*? What's next after finding a great pattern?

During this seminar, you will take a mathematical investigation from start to finish—from exploratory stages through reporting logically connected results—and you will find strategies that you can use with your students to develop their investigative skills.

You might also encounter some new mathematical facts and relationships, but the real purpose is for you to investigate *investigation.*

1. The Consecutive Sums Problem

Students in the first year course of the Interactive Mathematics Program (IMP) *are given three days to explore this lovely problem. We're really just going to scratch the surface today—if you're interested, you can take the opportunity to explore this problem in greater depth later.*

Take 10 to 15 minutes for a preliminary exploration of the problem below—just long enough to develop some initial conjectures.

Such a tiny amount of time is not nearly adequate for a thorough look at this investigatory problem, but even 15 minutes should give you a sense of what students begin to see as they explore it. For the moment, this glimpse is enough.

In the brief time you devote to the problem now, keep track of partial answers and any new questions that may come up.

PROBLEM

Positive integers go by many aliases: the counting numbers, the natural numbers, $\mathbb{Z}^+$.

1. The number 13 can be expressed as a sum of two consecutive counting numbers, $6+7$. Fourteen can be expressed as $2+3+4+5$, also a sum of consecutive counting numbers.

THE CONSECUTIVE SUMS PROBLEM:
Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can?

Experiment, look for patterns, and come up with some conjectures. Keep track of your observations below.

Remember: For now, take only 10 to 15 minutes.

Dissecting the problem

To investigate a problem well, you should get right to its heart. The first two sentences of problem 1 just say what is meant by "sum of consecutive counting numbers," and the last two sentences are merely guidance for the student. The problem's essence is in the middle two sentences:

> Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can?

Even this can be boiled down. The real information is:

> ... counting numbers ... expressed as sum of two or more consecutive counting numbers ...

2. Concealed within that deceptively simple boiled-down version are at least five essential features of the problem. Two are given to you. Find *at least three* others.

(a) It is about a *sum*.

(b)

(c)

(d)

(e) There are restrictions that the problem *could* make, but *does not*. The fact that it *fails* to make more restrictions is part of what makes it *this* problem and not another.

This, of course, is a feature of every problem. Learning to notice what is not *stated is extremely hard for everybody.*

Feature (e) may seem almost too silly to list, but it is important! For example, the problem refers to a "sum of two or more consecutive counting numbers." A more restrictive problem might ask "Which numbers can (or cannot) be expressed as a sum of exactly three consecutive counting numbers?"

This is a new *problem, and an interesting one!*

Similarly, the problem asks which numbers can be expressed at all, in any number of ways. A more specific problem might ask "Which numbers can (or cannot) be expressed in exactly one way (or two or ...) as a sum of consecutive counting numbers?"

This is another *new and interesting problem!*

Modifying the problem

Two great problems are listed in the previous paragraphs, but they are not the only good ones that come from changing features of the original problem.

3. By yourself or with others, brainstorm to see what related problems evolve from THE CONSECUTIVE SUMS PROBLEM as you change the features one (or at most two) at a time. Write down and share this set of new problems.

How do you decide, before investigating, which will be a worthwhile problem to pursue? Is it intuition? Experience? What goes into your decision?

4. Pick one or more of your problems and explore them just long enough to build some preliminary conjectures.

For now, take only 10 to 15 minutes. As before, you won't have enough time for a real investigation, but you should get a rough idea of what the problem has in store.

Problem-posing strategies

Problem 3 asked you to "change the features," but *how* should that be done? Are there any reliable ways to do that and get "good" problems as a result?

As you gain experience, you'll develop your own set of tricks for modifying the features of a problem, but here are four that are almost always among the most useful. These tricks can come in handy during class when you're trying to decide which direction to pursue next (or which challenge to give to kids who have finished early).

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i. Make a feature more restrictive: If the problem is about triangles, restrict it to right (or scalene or . . .) triangles. If the problem uses a calculation that involves two or more numbers, restrict it to *exactly* two (or three or ...).

This is sometimes referred to as finding special cases.

ii. Relax a feature: If the problem is about right triangles, see how it changes if you allow *all* kinds of triangles, or maybe all polygons. If the problem uses a restricted subset of numbers (e.g., only $\{1, 2, 3, \ldots\}$), see what happens when you expand that set in various ways.

This is sometimes referred to as generalizing*, or extending the domain.*

iii. Alter the details of a feature: If the problem concerns right triangles, see how it changes if you choose acute triangles. If the problem calls for one set of numbers (e.g., $\{1, 2, 3, \ldots\}$), try a different set (e.g., $\{1, 3, 5, 7, \ldots\}$ or $\{0, 3, 6, 9, \ldots\}$ or $\{0.5, 1, 1.5, 2, \ldots\}$). If the problem uses arithmetic operations, see what happens if you systematically alter them (e.g., substituting $+$ and $-$ for $\times$ and $\div$ or vice versa), and if it specifies equality, see what happens if you require a specific inequality (e.g., $>$).

These modifications may change the domain *of a problem, or alter a parameter.*

iv. Check for uniqueness: If the problem only asks *if* something can be done, ask if (or when) it can be done in *only one way*.

Asking "how many ways can this be done?" *is often productive.*

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5. Now, go back to the CONSECUTIVE SUMS PROBLEM. Look over the list of features you made for problem 2 and see if applying these rules to each of the features gives you any new problems.

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2. You've got a conjecture. Now what?

Intuition and mathematical taste

When students are asked to investigate, the ideas and difficulties that arise are inevitably less predictable than when the course is all laid out for them in advance. Some problems they pose lead to unanticipated treasures. Others seem likely to take time without giving the students much in return. Without having more than a few moments to think about the problems, you may find yourself in the position of having to decide which direction to take.

Problems 6–9:

Here are several pairs of variations on the original consecutive sums problem. Look at each pair, and try to decide, *without first pursuing the problems,* which choice seems more likely to lead somewhere.

"Without first pursuing the problems" does not mean that you cannot think about it at all. And "seems more likely to lead somewhere" does not mean "which will surely lead to interesting consequences."

6. (a) Which numbers can be expressed as sums of consecutive prime numbers?
(b) Which numbers can be expressed as sums of consecutive square numbers?

7. (a) Which numbers can be expressed in exactly seven ways as sums of consecutive counting numbers?
(b) Which numbers can be expressed in exactly one way as sums of consecutive counting numbers?

8. (a) Which numbers can be expressed as products of consecutive counting numbers?
(b) Which numbers can be expressed as differences of consecutive counting numbers?

9. (a) Which numbers can be expressed in exactly three ways as the sum of exactly three consecutive odd numbers?
(b) Which numbers can be expressed as the sum of consecutive odd numbers?

Reflect and Discuss

10. How did you make your decision in each of the previous problems? What "rules of thumb" did you use to help distinguish between problems that are probably good and ones that are probably not worthwhile?

And so do his sisters and cousins and aunts and ...

Even good problems can't all be pursued: too many problems, too little time. One way to decide among them is to think about which mathematical connections you most want to make.

Here, reminiscent of the TV game *Jeopardy!*, problems are listed by their answers, because the answers are one way of seeing potential connections to the rest of the curriculum.

PROBLEM

11. CONSECUTIVE SUM JEOPARDY:
In your investigations of the sisters, cousins, and aunts of the CONSECUTIVE SUMS PROBLEM, what questions (if any) have you run across which have the following sets of numbers as *answers* or partial answers? ∢

Here is an example of a sister-problem whose answer is on this list: "What numbers are produced as sums of exactly two consecutive counting numbers?"

(a) Only even numbers

(b) Only odd numbers

(c) Only prime numbers

(d) Only powers of two

(e) Only powers of three

(f) Only multiples of 3

(g) Only multiples of 5

(h) Only square numbers

(i) Differences of two square numbers

(j) Only triangular numbers

(k) Only differences of two triangular numbers

(l) Only factorials

(m) Only quotients of two factorials (permutations)

Following up a conjecture: Proof

This is your chance to pursue an investigation beyond the first few minutes.

At the beginning of an investigation, searching for a pattern is often a sensible thing to do. Unfortunately, students' investigations too often *end* when they've found what seems to be a pattern. Finding and describing an observed pattern is only the first step of an investigation. Next comes the essence of mathematical thinking: the effort to know, and show logically, that the pattern continues, why the pattern occurs, and how it logically follows from and connects with what is already known.

Elementary algebra is sufficient for solving these problems, but there are often ways of giving solid proofs—logical arguments, and not just appeals to pattern—that would work for fifth- or sixth- grade students who have never had algebra. See if you can find both *kinds of supports.*

Below are several conjectures people have made as they investigated the CONSECUTIVE SUMS PROBLEM (and its cousins). Find a proof or counterexample (an example that shows the statement is not always true) for each one. These problems form a path to a conclusion, so try to justify each statement.

12. The sum of two consecutive counting numbers (CCNs) is always odd.

13. The sum of three CCNs is a multiple of 3.

14. The sum of five CCNs is a multiple of 5.

15. The sum of *any* number of CCNs is a multiple of that number.

16. The sum of any odd number of CCNs is a multiple of that odd number.

17. Odd multiples of 3 (except 3 itself) can always be expressed in at least two ways as sums of CCNs.

18. Odd primes can be expressed in only one way as the sum of CCNs.

Complete, then prove the given statement.

19. The sum of an odd number of CCNs can be odd or even. It will be odd if . . .

Complete, then prove the given statement.

20. The sum of an even number of CCNs can be odd or even. It will be odd if . . .

21. The sum of a sequence of CCNs is a multiple of at least one of the numbers in the sequence.

22. The sum of a sequence of CCNs is a multiple of the mean of the first and last numbers in the sequence.

23. Any multiple of an odd number greater than 1 can be expressed as a sum of CCNs.

You may have already guessed where the preceding problems were leading. Even if you haven't, you should now be ready to put it all together.

24. The Consecutive Sum Theorem
The only numbers which can be expressed as a sum of CCNs are the multiples of an odd number greater than 1. That is, a counting number, N, is the sum of consecutive counting numbers if and only if N is not a power of 2.

If there's time, take a few minutes to organize a self-contained proof of the theorem.

Students often make conjectures that turn out to be false. As their teacher, it is important not only to help them see why the conjecture is false, but also to recognize what *correct* (but perhaps incomplete) observations led them to the false conjecture.

25. For each conjecture that you determined not to be true in problems 12–23, see if you can guess what *correct,* but incomplete, observations might have led to that conjecture.

Did thinking through and proving your conclusions to the statements in problems 12-23 *help you understand and prove the consecutive sums problem? Are these proofs similar or different from the proofs you encounter (or expect to encounter) in your teaching? If so, how?*