Abstract clotheslines

Here's a seemingly cockeyed idea: Take the distribution of sums for two dice and hang it out on a "clothesline" of powers of x:

$$x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$$

Note that the general term here is

$$\{\text{number of ways you can get } s \text{ as a sum}\} \times x^s$$

Why would you do such a thing? Well, it's just another code—another representation, if you will—of the information in the distribution table. It may even be easier to write down. It certainly takes less space.

So, now what? You can "do algebra" with this expression. For example, aren't you itching to factor it?

This activity is adapted from the Ways to Think About Mathematics: Activities and Investigations for Grade 6-12 Teachers, *developed at EDC and published by Corwin Press. The image of an expression as a clothesline is due to Herbert Wilf and is taken from his beautiful book,* Generatingfunctionology, *published by Academic Press.*

1. Using a *CAS* (like the TI-89) or paper and pencil, completely factor the 2-dice distribution polynomial

$$x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}.$$

Try factoring it by hand and see how far you get. Don't be shy about using a CAS to finish off the job.

One way or another, you should have found that the polynomial is equal to $x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2$. Now dig a little deeper into this factorization. It can't come out so nice without something going on, some hidden meaning, behind the scenes. Here's just one possible interpretation of the factorization:

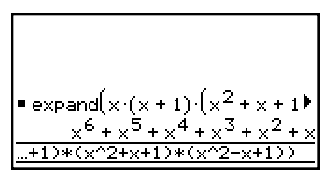
If not, take a few minutes to confirm this claim.

One thing formal algebraists do is hunt for hidden meaning that emerge from algebraic calculations. Could the power of 2 in the factorization have anything to do with the fact that we rolled 2 dice?

Notice that each factor is a square. That is, our original distribution polynomial factors as

$$\Big[x(x+1)(x^2+x+1)(x^2-x+1)\Big]^2$$

The insides may be too complicated to think about, so expand this product:



$$x(x+1)(x^2+x+1)(x^2-x+1) = x+x^2+x^3+x^4+x^5+x^6.$$

Hmm, that's surprising! We've just shown that

$$x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$
$$\text{equals} \qquad \left(x+x^2+x^3+x^4+x^5+x^6\right)^2.$$

Do you agree that $x+x^2+x^3+x^4+x^5+x^6$ is the distribution polynomial for the rolls of 1 die ?

Among other things, this says that the distribution polynomial for two dice is the square of the distribution polynomial for one die. Let's see if we can figure out why this *should* be the case. If we're lucky it will provide insight into the general situation (with n dice).

How, for example, would you get the coefficient of x^6 in the product? Of x^{10}? Of x^k?

2. Explain, as if you were working with a high school student, how you'd multiply out

$$\left(x+x^2+x^3+x^4+x^5+x^6\right)^2.$$

3. Explain why the expansion of

$$\left(x+x^2+x^3+x^4+x^5+x^6\right)^2$$

should give you the distribution for the sums that show up when rolling two dice.

4. **Does it work for 3 dice, too?**
If you expand the polynomial

$$(x+x^2+x^3+x^4+x^5+x^6)^3,$$

what do you think you'll get? Check to see if you're right!

PROBLEM

How could you use a CAS to find the distribution polynomial for the possible sums when you throw n dice (for a specific value of n)?

5. THE DICE SUM PROBLEM
Conjecture the value of the distribution polynomial for the possible sums when you throw n dice. Prove that your conjecture is correct.

Check Your Understanding

6. **Make a conjecture:** What is the most *likely* sum when n dice are thrown?

7. Test your conjecture for $n = 5, 6$, and 7, then check your answer using the polynoimals on page 3.

Distribution polynomials for dice problems (for 2–7 dice)

$$
\begin{aligned}
(x+x^2+x^3+x^4+x^5+x^6)^2 &= x^{12}+2x^{11}+3x^{10}+4x^9+5x^8+6x^7\\
&\quad +5x^6+4x^5+3x^4+2x^3+x^2\\
(x+x^2+x^3+x^4+x^5+x^6)^3 &= x^{18}+3x^{17}+6x^{16}+10x^{15}+15x^{14}+21x^{13}\\
&\quad +25x^{12}+27x^{11}+27x^{10}+25x^9+21x^8\\
&\quad +15x^7+10x^6+6x^5+3x^4+x^3\\
(x+x^2+x^3+x^4+x^5+x^6)^4 &= x^{24}+4x^{23}+10x^{22}+20x^{21}+35x^{20}+56x^{19}\\
&\quad +80x^{18}+104x^{17}+125x^{16}+140x^{15}\\
&\quad +146x^{14}+140x^{13}+125x^{12}+104x^{11}80x^{10}\\
&\quad +56x^9+35x^8+20x^7+10x^6+4x^5+x^4\\
(x+x^2+x^3+x^4+x^5+x^6)^5 &= x^{30}+5x^{29}+15x^{28}+35x^{27}+70x^{26}+126x^{25}\\
&\quad +205x^{24}+305x^{23}+420x^{22}+540x^{21}\\
&\quad +651x^{20}+735x^{19}+780x^{18}+780x^{17}\\
&\quad +735x^{16}+651x^{15}+540x^{14}+420x^{13}\\
&\quad +305x^{12}+205x^{11}+126x^{10}+70x^9\\
&\quad +35x^8+15x^7+5x^6+x^5\\
(x+x^2+x^3+x^4+x^5+x^6)^6 &= x^{36}+6x^{35}+21x^{34}+56x^{33}+126x^{32}+252x^{31}\\
&\quad +456x^{30}+756x^{29}+1161x^{28}+1666x^{27}\\
&\quad +2247x^{26}+2856x^{25}+3431x^{24}+3906x^{23}\\
&\quad +4221x^{22}+4332x^{21}+4221x^{20}+3906x^{19}\\
&\quad +3431x^{18}+2856x^{17}+2247x^{16}+1666x^{15}\\
&\quad +1161x^{14}+756x^{13}+456x^{12}+252x^{11}\\
&\quad +126x^{10}+56x^9+21x^8+6x^7+x^6\\
(x+x^2+x^3+x^4+x^5+x^6)^7 &= x^7+7x^8+28x^9+84x^{10}+210x^{11}+462x^{12}\\
&\quad +917x^{13}+1667x^{14}+2807x^{15}+4417x^{16}\\
&\quad +6538x^{17}+9142x^{18}+12117x^{19}+15267x^{20}\\
&\quad +18327x^{21}+20993x^{22}+22967x^{23}+24017x^{24}\\
&\quad +24017x^{25}+22967x^{26}+20993x^{27}+18327x^{28}\\
&\quad +15267x^{29}+12117x^{30}+9142x^{31}+6538x^{32}\\
&\quad +4417x^{33}+2807x^{34}+1667x^{35}+917x^{36}\\
&\quad +462x^{37}+210x^{38}+84x^{39}+28x^{40}+7x^{41}+x^{42}
\end{aligned}
$$

Variable–free distribution polynomials

Mathematicians usually refer to distribution polynomials as generating functions. *That is, a generating function for a sequence (finite or infinite) is its corresponding distribution polynomial (or series).*

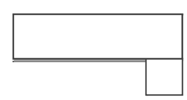
For example, the generating function for row 5 of Pascal's triangle is $1+5x+10x^2+10x^3+5x^4+x^5$ *(the coefficient of* x^k *is entry* k *in row 5 of the triangle, a.k.a.* $\binom{5}{k}$*).*

Depending on the courses you teach, perhaps your reaction to these is, "This is all very interesting, but what if my students don't know about polynomials?" Remember, a distribution polynomial is just a bookkeeping device—a clothesline on which to hang your terms. More to the point, sometimes you can translate algebraic symbolism into something younger kids can understand. Fortunately, this is one of those times. In fact, the method used here is useful even for those with an understanding of algebraic manipulation and symbolism, for it will allow you to derive a proof of your conjecture concerning the most likely sum when n dice are thrown.

Here's a gadget that lets you figure out how likely a certain sum is if you roll several dice. It requires some setting up, so bear with us for a bit. Imagine you have a row of numbers consisting of zeros *except* for six 1's somewhere in the "middle":

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0

You also have a little "viewer" that is six numbers wide which you can slide along your row.



The viewer has an extra "drop" compartment at the bottom right which is used to create the numbers in the next row. If you place the viewer on a given row, you record the sum of the numbers in the box in the drop compartment (as shown):

[0 0 0 0 0 0] 1 1 1 1 1 1 0 0 0 0 0 0 0 0
[0]

Now slide the viewer along the top row, filling in the second row:

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 1

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 1 2

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 1 2 3

⋮ ⋮ ⋮

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 1 2 3 4 5 6 5 4 3

Here's the whole second row:

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 2 3 4 5 6 5 4 3 2 1 0 0 0

Then start the third row using the same process:

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 2 3 4 5 6 5 4 3 2 1 0 0 0
0

0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 2 3 4 5 6 5 4 3 2 1 0 0 0
0 1 3

0	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
0	**0**	**0**	**0**	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**5**	**4**	**3**	**2**	**1**	**0**	**0**	**0**
					0	**1**	**3**	**6**											

0	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
0	**0**	**0**	**0**	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**5**	**4**	**3**	**2**	**1**	**0**	**0**	**0**
					0	**1**	**3**	**6**	**10**										

Note also that the table has been truncated due to space limitations. Take a few minutes to insert the remaining terms of rows 3, 4 *and* 5.

Confirm that rows 1, 2 *, and* 3 *agree with the distribution table for sums when* 1, 2 *, and* 3 *dice are rolled.*

And eventually, you get the following table (each row starts out with a stream of zeros): ⊳

1	1	1	1	1	1	0	0	0	…					
1	2	3	4	5	6	5	4	3	2	1	0	0	0	…
1	3	6	10	15	21	25	27	27	25	21	15	10	6	…
1	4	10	20	35	56	80	104	125	140	146	140	125	104	…
1	5	…												
⋮	⋮													

Why should the sum of 6 consecutive entries from the 2-dice sum table equal an entry from the 3-dice sum table?

Did you notice that the nonzero entries in the first three rows are identical to the frequency tables you made for 1, 2, and 3-dice sums? Use the distribution polynomial for the 4-dice sums to show that what you can see of the fourth row matches, too. Is this a coincidence? If not, why does the viewer provide the distribution tables, too? Think about it!

