Geometer's Sketchpad:
A tool for conceptual understanding and mathematical exploration.

Geometer's Sketchpad is a dynamic geometry software application that can be used to teach, reinforce, and explore many mathematical concepts in geometry *and* algebra.

These first activities are designed to introduce you to some of the features *Geometer's Sketchpad (GSP)* has to offer. You might find it useful to take notes in the space provided or on a separate sheet of paper. However, some of the best features of this version of GSP are the Help pages, which are available by choosing **Contents** from the **Help** menu at the top right of the page.

1. Use the compass tool to construct a circle in the middle of the screen.

2. Use the **selection tool** to grab the point on the circle and observe what happens as you move the point. *To grab an object, place the cursor over the object, then click and hold the (left) mouse button.*

3. Now, use the **selection tool** to grab and move the center of the circle. How is what happens the same or different than what happened in 2?

4. Use the **selection tool** to grab the circle. Is this similar to grabbing the point on the circle or the center?

5. Move the circle to the upper left corner of the screen.

6. Use the **straightedge tool** to construct a segment in the middle of the screen. *To create a segment, select the* ***straightedge tool*** *– being sure it is set to segment, rather than line or ray (to change the selection, click and hold on the straightedge tool, move the cursor to the appropriate selection, then release the mouse button. Next, click on the screen to construct one endpoint, then click on another location on the screen to construct the other endpoint (and the segment).*

7. Choose the **selection tool** (*you can always get back to the selection tool by hitting the* **esc** *key*) and then click the mouse on an empty space on the page. This makes sure that no item is selected.

8. Place the cursor over the segment you create and click the mouse. *Note that the segment becomes "highlighted."*

9. Click on one of the endpoints of the segment. *Note that the segment is still highlighted even after you highlighted the endpoint.*

10. Click on the endpoint again and note that the point is no longer highlighted.

11. Being sure that the segment is still highlighted, open the **Construct** menu by clicking on **Construct** at the top of the page. *Notice that the only two options that are available are "Point on Segment" and "Midpoint" (the others appear lighter).*

12. Choose **Midpoint** by moving the cursor down to "Midpoint" and clicking the mouse button. *Note that this can also be done by typing* - M *(that is, typing "m" while holding down the Apple key,) on a Macintosh and* ctrl – m *on a Windows machine.*

13. The midpoint of the segment should now be highlighted. Use the selection tool to select the segment again and "deselect" the midpoint (by clicking on them). You could also deselect all objects (click on an empty space on the sketch), then click on the segment.

14. Choose the **Measure** menu at the top of the page and select **Length** (the only other options when a segment is highlighted are **Slope** and **Calculate**). *Note that the length of the segment appears on the screen.*

15. You can use the **Measure** menu to measure other quantities as well. Select the circle (and nothing else) and have GSP compute the area and circumference of the circle. Are there any other measurements that can be made with the objects you've created? ***Try it!***

16. What happens to these measurements when you alter the objects? ***Try it!***

    Remember, you can always choose **Undo** *from the* **Edit** *menu if you move or delete something accidentally.*

Triangles

1. Create a new sketch by choosing **New Sketch** from the **File** menu.

2. Use the **segment tool** to construct a triangle. *First construct a segment, then another segment with "starting point" at one of the endpoints of the original segment. Note that when using the segment tool, a segment's endpoint "grows" when the mouse cursor is directly over it, telling you that if you click, you will begin at that endpoint.*

3. Now, use the **Label tool** (pointing finger) to label the three vertices of the triangle you just constructed.

4. Use the **Selection tool** to highlight the vertices in the following order: First **A**, then **B**, then **C**. *Be sure that nothing else is highlighted. If that's not the case, click on an empty space (to "un-highlight" everything) then select the three vertices again.*

5. Choose **Angle** from the **Measure** menu. *What angle got measured?*

6. Have Sketchpad measure the other two angles of triangle ABC for you. *In what order should you select the vertices in order to measure angle* BCA*? Angle* BAC? *What about angle* CAB? *In what way does order matter?*

7. Use the **selection tool** to grab one or more of the vertices or edges of the triangle. *What happens to the angle measurements as the objects move?*

8. What is the sum of the three angle measurements you measured? Is the sum what you *expect* it to be? What happens to this sum when you move the vertices?

9. *Now, we'll have Sketchpad compute the sum for us.* Choose **Calculate** from the **Measure** menu. Notice that a calculator "pops up".

10. Click on one of the angle measurements from your triangle, then either click on the "+" on the popup calculator or type **+**. *Note that the calculator window will display what you are entering.*

11. Click on another angle from your triangle, then **+**, then the third angle. Click on the **OK** button on the calculator to compute the sum. *What do you get? Is it what you expected? Is it what you got when yon calculated "by hand"?*

Other polygons

1. Use GSP to construct a quadrilateral and compute the sum of the measurements of its angles. Is the situation the same as it was with triangles? Do you *always* get the same sum? If so, what is the common sum?

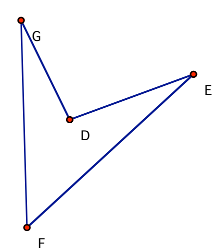
2. Repeat problem 1, this time with a pentagon.

3. Repeat problems 1 and 2 with a hexagon.

4. Make a conjecture about the angle sum of an octagon. Use GSP to check your conjecture.

5. What is the angle sum of a decagon (a 10-sided polygon)?

6. Have you figured out a way to make the formula fail? For what type of polygons does the formula fail to work? *Try to make one of the angles greater than 180 degrees as in the figure on the right. What happens to the angle sum? What's going on?*



As it turns out, in order for the formula to work with Sketchpad, the polygon must have the property that if you connect *any* two points on or inside the polygon (that is, on the interior or an edge) with a segment, then the whole segment must lie on or inside the polygon. Such a polygon is called *convex*. Forget about GSP for a second. Does the actual formula still work when the polygon isn't convex?

Back to some GSP features

1. Create a new sketch and construct a segment.

2. Use the **selection tool** to grab and move the segment's length so that it appears just above the segment on the page.

3. Use the **selection tool** to make sure that the segment and both endpoints are the only highlighted objects. *This can also be done by holding down the mouse button on an empty part of the page near the objects and creating a box around the objects (all objects in the box will become highlighted).*

4. Select **Perpendicular Lines** from the **Construct** menu. What happens?

5. How can you construct the perpendicular bisector of the segment? ***Try it!***

Now, I'd like to clean up the picture by removing some of the extraneous objects.

6. Use the **selection tool** to select one of the endpoints of the segment, being sure that nothing else is highlighted.

7. Choose **Cut** from the **Edit** menu (alternatively, type - x on a Mac or ctrl – x in Windows). What happened?

8. Since a little too much got eliminated there, choose **Undo** from the **Edit** menu (you can also type - z on a Mac or ctrl – z in Windows).

9. Now, select an endpoint of the segment (and nothing else) and select **Hide** from the **Edit** menu. What happened this time?

10. Do you remember the "compass and straightedge" construction of the perpendicular bisector? ***Let's try it!***

Constructions versus drawings

1. Open a new sketch by selecting New Sketch from the File menu.
2. Draw a rectangle using the straightedge tool (set to segment). *Note that points "expand" when the cursor is on top of them, allowing you to be sure that you're "connecting the dots"*.

3. What happens if you grab and move one of the vertices? Do you still have a rectangle?

4. Using the text tool, click on the vertices one at a time. What happens?

5. Now, using the selection tool, select points *A, B,* and C, in that order, then choose "Angle" from the Measure menu. What happens?

6. Now, have *GSP* measure the other 4 angles. What *should* the sum of the angles be? Is the sum what you expect?

7. Choose Calculate from the Measure menu. A small calculator should appear on the screen.
8. Be sure the calculator does not block your view of the quadrilateral by grabbing the top of its window and moving it, if necessary.
9. Click on the measure of one of the 4 angles, then click on the "+" on the calculator, then click on the measure of another angle, then click "+", and so on until you have added all 4 angles on the calculator. Finally, click "OK".
10. Does the angle sum given by the calculator agree with the one *you* computed earlier?

Now, we'll *construct* a rectangle.

11. Open a new sketch by selecting New Sketch from the File menu.
12. Construct a segment and the lines perpendicular to the segment through the two endpoints of the segment.

13. Use the point tool to construct a point on one of the two perpendicular lines.

14. Construct the line through this new point that is parallel to the original segment. *You'll need to highlight the point and segment*.

15. Use the select tool to construct the fourth vertex of our rectangle by clicking on the intersection of the lines at which the point lies.

16. Hide the lines (but not the points or segment).

17. Use the straightedge tool to construct the remaining three sides of the rectangle.

18. Use the selection tool to grab and move various parts of the rectangle. When does the rectangle move and when is it stretched or shrunk? Try out various combinations.

Special Points on a triangle:

1. Open a new sketch by selecting New Sketch from the File menu.

2. Construct a triangle using the straightedge tool.

3. Construct the perpendicular bisector of *two* of the triangle's edges, as well as the intersection of these two lines.

4. Construct the perpendicular bisector of the third edge of the triangle. What does the relationship seem to be between the three bisectors?

5. Move the vertices of the triangle. Does the relationship still seem to hold? Can you prove that it *always* does?

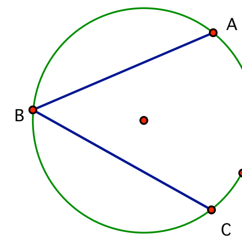
6. Construct the three angle bisectors of the triangle. What relationship do they seem to have? Can you prove it?

7. Construct the three medians of the triangle (segment connecting vertices to the midpoint of the opposite side). What relationship do they seem to have? Can you prove it?

And now for something completely different

1. Construct a circle on a new sketch

2. Construct an inscribed angle on the circle by creating two chords sharing a common point. Label the three points *A, B,* and *C* with *B* as the vertex of the angle.



3. Using the selection tool, highlight *A* and *C* (in that order) and the circle, then choose "Arc on Circle" from the Construct menu.

4. Open the Display menu, slide the cursor down to Color (to open its menu) and then select a different color from the color of the circle. Which arc was constructed?

5. Now, use the selection tool to deselect everything, then select *C* then *A* (in that order) and the circle and select "Arc on Circle" from the Construct menu. Which arc is constructed this time?

6. Change the color of this arc so that it differs from the color of the circle and the color of the first arc.

7. Use the selection tool to highlight the minor arc (the smaller one) and choose "Arc Length" from the Measure menu.

8. Measure angle *ABC* by highlighting *A, B,* and *C* (be sure that *B* is in the middle) and selecting "Angle" from the measure menu.

9. How do the two measurements compare as you move any of *A, B,* and *C*?

10. Does the relationship you found *always* hold? Can you prove it?

11. Construct a new circle on your sketch.

12. Use the straightedge tool to construct a **line** containing two points on the circle and label these points *A* and *B* (to change a label, double-click on it using the text tool and type in the new label). *Note that when the cursor is over the circle, the message "…from point on circle …" appears at the bottom right of the screen.*

13. Construct another line containing two different points on the circle. Label these points *C* and *D*.

14. Construct the intersection of the two lines. Label the intersection *X*.

15. Use the selection tool to move the points around and observe which objects move.

16. Measure the lengths *AX, BX, CX,* and *DX*, then compute the products $AX \cdot BX$ and $CX \cdot DX$. How do these products compare when the points are moved around? In particular, does it matter whether *X* is inside or outside the circle?

17. Can you prove that the relationship you observed *always* holds?

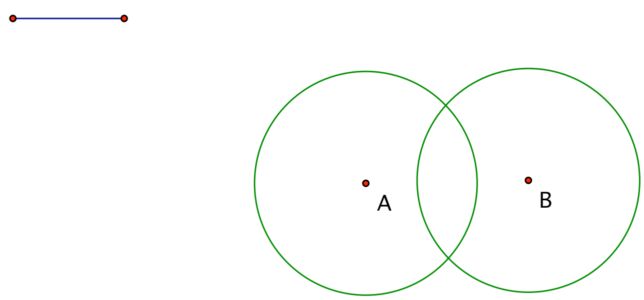
Some additional explorations

Try some of the following ‘problems, using *Geometer’s Sketchpad*.

1. We know that the set of points that are equidistant from two points in the plane is the perpendicular bisector of the segment whose endpoints are the two points. But what if we *didn’t* know that? Here’s a possible way to discover that fact.

   Plot two points in the middle of the screen and construct a segment at the top left portion of the screen.

   Construct circles centered at the two points each having a radius congruent to the segment you constructed. *You can do this by highlighting the two points (points A and B below) and the segment (without the endpoints), then selecting “Circles by center and radius” from the* **Construct** *menu. (Note that we could have also Measured the Length of the segment and highlighted the length calculation instead of the segment itself)*



   Construct the intersection points of the two circles (if necessary, first grab the right endpoint of the segment and drag it to increase the length of the radii so that the circles intersect). Notice that since the two circles have congruent radii, the intersection points will always be the same distance from A as they are from B.

   Highlight the intersection points of the circles and select “Trace Intersections” from the Display menu.

   Now, grab the right endpoint of the segment and *slowly* drag it back and forth (as far as you can make it go) to see all of the points that are equidistant from A and B. *If all went well, the points should trace out the perpendicular bisector of A and B. To check this, construct the perpendicular bisector of A and B (by constructing* $\overline{AB}$*, its midpoint, and the perpendicular line to* $\overline{AB}$ *through the midpoint) and check!*

If you want to get rid of the trace, choose “Erase Trace” in the **Display** menu. In order to *stop* having *Sketchpad* tracing the intersection points, highlight these two points (and nothing else) and select “Trace Points” from the **Display** menu again (this will “uncheck” Trace Points and it will be inactive until you ask it to trace another point (or points).

The following problems are related to the one on the previous page. Think about how you could adapt the previous method (or try another method). As before, suppose A and B are two points in the plane.

2. What do you think the set of points that are twice as far from A as they are from B will look like? (A line, a segment, a ray, a parabola, a circle, an ellipse, …?)

   A quick way to be able to construct segments so that one is twice the length of the other is to create the larger segment and the subsegment from one of its endpoints to its midpoint.

3. Would the set of points that are 3 times as close to A as they are to B have the shape as the figure in problem 2?

4. What if the points are *n* times the distance from A as from B?

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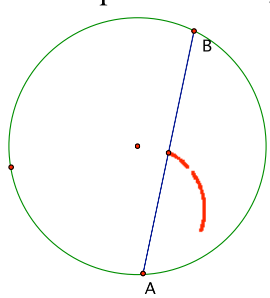
5. Construct a circle, two points on the circle (neither should be on the of the parents of the circle), and the chord between the points.

   Next, construct the midpoint of the chord and, while the midpoint is still highlighted, choose "Trace Midpoint" from the **Display** menu.

   Now, keeping one of the endpoints of the chord stationary, grab the other endpoint and drag it about 1/4 of the way around the circle.



What do you think the resulting "locus" will look like? Be as specific as possible. Don't just describe the general shape, describe some specific properties (For example, if you think you get a circle, specify its center and radius (or diameter)). Continue dragging the point around the circle to test your conjecture

6. Repeat problem 5, but this time choose the point that is 1/4 of the way from the leftmost endpoint of the segment and the rightmost endpoint. *Guess the shape (and its specific features) and check your guess.*

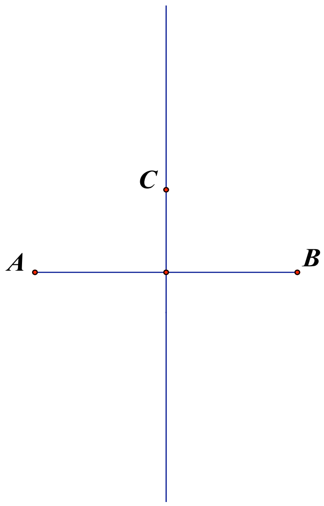
7. Repeat problem 5 (and 6) but just place an *arbitrary* point on the chord. *Note that the ratio of the distance from the point to one of the endpoints of the chord and the length of the chord remain constant.*

Can you justify your conjecture more "rigorously" than observing that it seems to be right?

Construct a line segment and label the endpoints of your original segment *A* and *B*

Construct the perpendicular bisector of (using the **Construct Midpoint** and **Construct Perpendicular** commands)

Now, construct a point on the perpendicular bisector of $\overline{AB}$ and label it *C*.



What do you notice about the distances *AC* and *BC*? If you move stuff around, does this relationship remain?

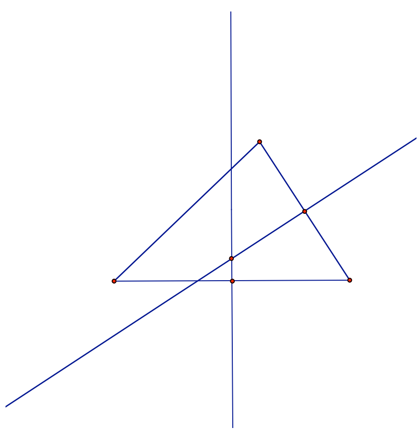
Why must this relationship *always* hold?

Do you think the relationship is reciprocal? That is, if a point *D* satisfies the relationship (for *AD* and *BD*), then does *D* have to be on the perpendicular bisector of $\overline{AB}$?

Why must this relationship *always* hold?

Construct a triangle using the segment tool.

Construct the perpendicular bisectors of *two* of the sides of the triangle and label their intersection *X*. Then, hide the two perpendicular bisectors by highlighting them (and only them) and choosing **Hide Perpendicular Lines** from the **Display** menu.



Construct the perpendicular bisector of the third side of the triangle. What do you notice? Move the vertices of the triangle and see if the relationship still seems to hold.

Why must it *always* hold?

Investigate: When do the perpendicular bisectors meet inside the triangle? Outside the triangle? On the triangle? Be as specific as you can be.

Definition: The intersection of the perpendicular bisectors of the sides of a triangle is called the ***circumcenter*** of the triangle.

Construct the circle centered at the circumcenter of your triangle and going through one of the vertices of the triangle. What do you notice?

Last time, we confirmed that the perpendicular bisectors of the three sides of a triangle intersect at a single point, called the *circumcenter* of the triangle. The circumcenter is the center of the circle (called the *circumcircle* of the triangle) that contains the three vertices of the triangle.

Today, we're going to investigate whether this common intersection property works for other lines connected to the triangle.

Investigate the following questions with Geometer's Sketchpad *and* work to confirm your conclusions.

1. Do the angle bisectors of the triangle intersect at a single point?

2. Do the medians (segment connecting a vertex to the midpoint of the opposite side)?

3. Do the altitudes (through a vertex and perpendicular to the opposite side)?

The *circumcenter* is the center of a triangle's circumcircle. It can be found as the intersection of the perpendicular bisectors.

The geometric *centroid* (center of mass) of the polygon vertices of a triangle is the point which is the intersection of the triangle's three triangle medians (Johnson 1929, p. 249; Wells 1991, p. 150). The point is therefore sometimes called the median point. The centroid is always in the interior of the triangle.

The intersection of the three altitudes of a triangle is called the *orthocenter*.

The *incenter* is the center of a triangle's incircle. It can be found as the intersection of angle bisectors, and it is the interior point for which distances to the sides of the triangle are equal.

1. Use Geometer's Sketchpad to construct a triangle and its circumcenter, centroid, orthocenter, and incenter (but hide all of the stuff that led you to these points).

2. Move the vertices of your triangle, keeping an eye on the 4 special points. What do you notice?

3. Does any group of more than two points seem to be collinear?

4. For what kinds of triangle will the circumcenter be the same as one or more of the other points? For what kind of triangle are all 4 points the same?

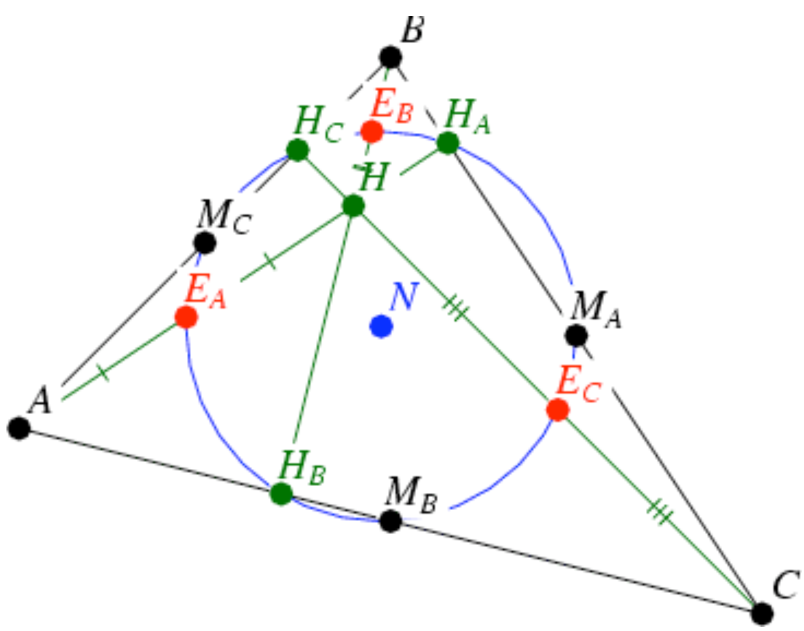
The *Euler Line* of a triangle is the line on which the orthocenter, triangle centroid, circumcenter, nine-point center, and number of other important triangle centers lie.

1. Does the incenter *ever* seem to lie on the Euler Line?

2. Construct the midpoint of the segment with endpoints at the circumcenter and orthocenter. Label this point *N*.

3. Construct the circle centered at *N* going through the midpoint of one of the sides. Does it matter which midpoint you choose? Hmm, isn't that interesting?

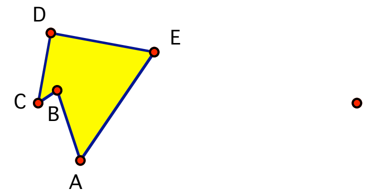
The nine-point circle, also called Euler's circle or the Feuerbach circle, is the circle that passes through the feet of the perpendiculars H_A, H_B, and H_C dropped from the vertices of any triangle *ABC* on the sides opposite them. Euler showed in 1765 that it also passes through the midpoints M_A, M_B, and M_C of the sides of *ABC*. By Feuerbach's theorem, the nine-point circle also passes through the midpoints E_A, E_B, and E_C of the segments that join the vertices and the orthocenter *H*. These points are commonly referred to as the Euler points. These three triples of points make nine in all, giving the circle its name. Its center, *N* (the nine-point center), is the midpoint of the line between the circumcenter and orthocenter.



Political Geometry: Putting the "spin" on "flip flops"

In this activity, we'll investigate transformations with *Geometer's Sketchpad*.

1. Construct a non-regular pentagon using the segment tool and use the point tool to construct a point that does *not* lie on the pentagon.

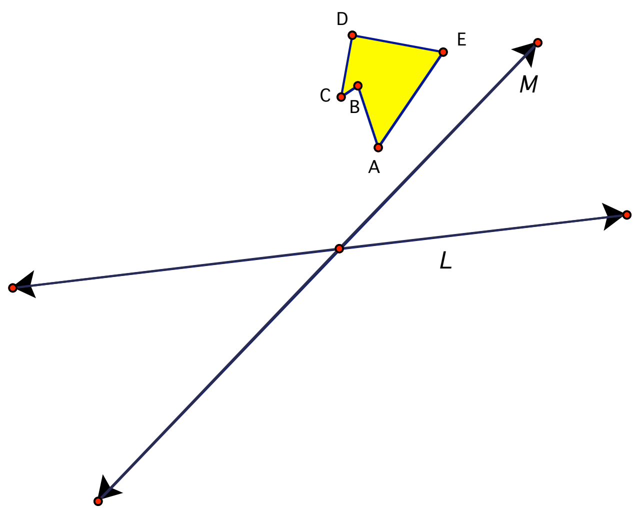


2. Double-click on the point to mark it as the center of a rotation (alternatively, you can select the point then choose **Mark center** from the **Transform** menu).
3. Now, highlight the pentagon and choose **Rotate…** from the **Transform** menu. When the dialog box opens, type "30" when prompted for an Angle. What happened?
4. You can also use the **selection tool (rotate)** to rotate the pentagon any angle if the pentagon is highlighted. Try it!
5. Move the point you marked as the center of rotation and observe what happens to the rotated pentagon. Then, move the original pentagon and observe the effect.
6. Now, highlight the pentagon and choose **Translate…** from the **Transform** menu.
7. When the dialog box appears, choose "Polar" and type in an angle and distance. Observe what happens. What is Sketchpad doing?
8. Repeat steps 6 and 7, but this time choose "Rectangular" and choose a Horizontal and Vertical distance to translate. What is Sketchpad doing here?
9. Next, create a segment on the screen and highlight the two endpoints then choose **Mark vector** from the Transform menu.
10. Highlight the pentagon and choose **Translate…** from the **Transform** menu. What happens?
11. Select one of the endpoints of the segment that defined the translation and move it. What effect does this have on the translated pentagon?
12. Select the pentagon and choose Copy from the Edit menu (or type ctrl-c).
13. Open a new sketch by typing ctrl-n or choosing New Sketch from the File menu.
14. Choose Paste from the Edit menu (or type ctrl-v).
15. Construct a segment with the **straightedge tool** then choose the **select tool** and double-click on the segment to mark it as a mirror (or highlight the segment, then choose **Mark mirror** from the **Transform** menu).
16. Highlight the pentagon and choose **Reflect…** from the **Transform** menu.
17. Move the mirror and/or move the original pentagon. What's the effect?

Now, we'll look at multiple transformations (one after the other)

18. Construct two **intersecting** lines (with no specific orientation - we'll move them around, eventually) and label them ***L*** and ***M***.

19. Construct a pentagon (again, without any specific properties so we can alter its appearance) and label the vertices ***A, B, C, D,*** and ***E***. We're going to reflect pentagon ***ABCDE*** over line ***L,*** then reflect the resulting pentagon over line ***M***. In both cases, we'll use the **Reflect** command in the **Transform** menu.



20. In order to do this, we need to "mark" the mirror and highlight the object being reflected. First, to mark line ***L*** as the mirror, **either** double-click on line ***L*** (if you're successful, line ***L*** will appear to vibrate briefly) or highlight line ***L*** and choose **mark mirror** from the **Transform** menu. After marking the mirror, highlight pentagon ***ABCDE*** and select **Reflect**. You should now see the desired reflection. Use the **text** tool to display the vertices, being sure that the reflected image of ***A*** is labeled ***A'***, etc. Move one or more of the vertices of ***ABCDE*** and notice that the reflection changes with any changes to the original figure or mirror.

21. Now, repeat the above process, this time marking line ***M*** as the mirror and highlighting pentagon ***A'B'C'D'E'*** before selection **Reflect** from the **Transform** menu.

22. Show the labels of the new reflection and be sure that you've got pentagon ***A''B''C''D''E''***. Now, hide (or ignore) pentagon ***A'B'C'D'E'***. How could we have transformed ***ABCDE*** *to* pentagon ***A''B''C''D''E''*** directly? (Hint: it's a rotation, but where does the center of rotation lie and what's the angle of rotation?)

23. Grab one of the points defining one of the mirrors and move it, thus changing the angle at which the mirrors intersect. What effect does this motion have on your conjectured angle of rotation? Using the **label/text** tool, **make a conjecture by filling in blank in the following sentence:** ***When the angle between the mirrors increases, the apparent angle of rotation _______.*** It might help to display the values of the angle between the mirrors and apparent angle of rotation for reference.

24. Check your conjectures by first double-clicking the point you think is the center of rotation (or highlighting the point and selecting **mark center** in the **Transform** menu). Then highlight pentagon ***ABCDE*** and select **Rotate** in the **Transform** menu. A dialog box will appear in which you are asked what angle you'd like to rotate the pentagon around the center of rotation. Type in your angle, keeping in mind that a positive angle will go "counterclockwise" and a negative angle will cause a clockwise rotation.

25. Does the result of the rotation end up on top of pentagon ***A''B''C''D''E''***? If so, move pentagon ***ABCDE*** and lines ***L*** and ***M*** to ensure that the your rotation really is the same as the double reflection. If your conjecture seems to hold true, go on to #28. If your conjecture did not turn out to be true, guess again and repeat #26.

26. **Take a few minutes to add to your conjecture. *Carefully* write out the result of your work. That is, spell out your conjectures, being sure to specify the precise location of the center of rotation and the relationship between the angle of rotation and the angle between the mirrors, including some reasons that you feel your conjecture should be correct.**

27. Now, move line ***L*** by "grabbing" one of the points which define ***L***. (You should be changing the angle at which ***L*** and ***M*** intersect. You'll probably notice that the rotated image no longer matches with the result of the double reflection. That's because, as you noticed above, the angle of rotation depends on the orientation of the two mirrors, ***L*** and ***M***. So, we need to start over, right? Well, not necessarily. Select the **Edit** menu and see if you can select **undo translate point**. If so, you'll be able to undo the change you just. It might also be a good idea to select **undo construct rotated objects**, too, so we're not distracted by the rotated image of our pentagon.

28. We'll now use the **Rotate** command a little differently. The command also allows us to rotate via a *marked* angle. If you know what angle you want to rotate (∠***AXA''***, perhaps? - where ***X*** is our center of rotation), you can highlight it (in the order ***A, X, A''***) and select **mark angle ...** from the **Transform** menu. This all assumes that you've already marked the center of rotation. The nice thing about this method is that if you change the mirrors (and therefore ***A''***), it also changes the angle of rotation, so if you measure this angle, you can keep track of the relationship between the angle of rotation and the angle between the mirrors.

29. **Now, write your new conjecture concerning the value of the angle of rotation (as a function of the angle between the mirrors). Don't change your original conjecture - add this conjecture to your original.**

30. Now start a new sketch. The question I'd like to consider now is ***what if the mirrors don't intersect?*** When we chose lines ***L*** and ***M*** above, we insisted that the lines intersected, but what happens if the lines are parallel? In the time remaining, investigate the result of a double reflection over two parallel lines. What familiar isometry appears to result? How does the distance between the two lines affect the resulting isometry?

Additional questions to consider

- Can we verify that our conjecture concerning the double reflections is correct? Actually, it's not as hard as it might appear. Think carefully about what property must be satisfied by the center of rotation (as it relates to the result of the rotation) and consider the angles involved and try to verify your conjecture.
- Can we do our original double reflection "in reverse"? That is, if I give you a rotation with a given center and angle of rotation, can you find two mirrors so that the result of the double reflection over the two mirrors is the same as the result of the rotation?
- Does it matter in what order two isometries are "combined"? That is,
- if we reflect over line ***L***, then reflect the image over line ***M****,* is this identical to the result of reflecting over line ***M***, then line ***L***?
- if we combine a reflection with rotation, does the order matter?
- What if you combine two rotations, two translations, or a translation and a reflection, or a translation and a rotation?
- What's the result of a **triple** reflection?

- How can we use *GSP* to create "Escher-type" tesselations?

More reflections

We've already looked into the result of a double reflection (whether the mirrors intersect, or not). Now, we're going to investigate the result of triple reflections, double rotations, and double translations.

1. Construct three lines (labeled *L, M,* and *N*) which can move independently. Orient them so that all three lines intersect, but not at a single point.
2. Construct a polygon, reflect it over line *L,* reflect the resulting polygon over line *M*, and reflect this polygon over line *N*.
3. In order to see better what's going on, **Hide** the "intermediate" polygons (keeping the original and final polygon). **Earlier, you may have conjectured that the result of this triple reflection is itself a reflection. Does this seem to be the case? If so, where does the mirror appear to be? Can you find a relationship between the original mirror and the new one you just found? If not, how can you tell? Is the resulting isometry a rotation or translation? Recall that in order to show that an isometry is a reflection, you need to find the mirror, for a rotation you need to specify the center and angle of rotation, and for a translation, you need to specify a vector.**
4. Now, move one of your lines so that the three lines appear to intersect at a single point. **How does the conjecture look now? Answer the same questions as in #3.**
5. To double check your conjecture from #4, **Open** a **New Sketch**, this time constructing the three lines so that they intersect at a single point. The best way to do this might be to construct 2 arbitrary intersecting lines, construct the point at the intersection of these two lines, and construct a new line "starting" at this intersection point. **Perform a triple construction and check whether your conjecture seems to be correct.**
6. Now, check to see what the result is when you perform a *double rotation*. Be sure to consider a number of cases (the two rotations have different or same centers, different or same angles). You'll probably notice (depending on the particular centers and angles you chose) that the resulting isometry seems to be a rotation. **Find the center and angle of rotation and try to determine whether there's a relationship between the two original centers and angles of rotation.** It will probably help to have *GSP* **measure** the various angles involved.
7. I've heard a claim that the composition of two rotations is either a rotation, translation, or the identity transformation. We can get the identity transformation if our second rotation is ρ^{-1}, where ρ is the first rotation, but how can we get a translation? Think about this before going on to the next step.
8. Choose one of the vertices of your polygon and mark it as the center of your rotation. Now, rotate the polygon around this chosen center using whatever angle you choose. Next, select another vertex of the polygon and rotate the resulting polygon around this new center, but with the **opposite** angle. **Check that the result of this double rotation is, in fact, a translation. What is the relationship between the direction and length of the translation and the two rotations?**

9. Perform a similar investigation concerning the composition of a reflection and a rotation. Is the result a recognizable isometry? **Explain your observations and conjectures.** Consider both cases $\mu \circ \rho$ and $\rho \circ \mu$ (μ and ρ are the reflection and rotation) and whether or not the center of rotation is on the mirror.

10. Now, write up <u>all</u> of the observations and conjectures you made in items 1-9 carefully and completely.

Finally, answer the following questions:

11. How can you tell whether a given isometry is a rotation? How can you determine the location of the center and value of the angle of rotation? **Provide step-by-step instructions.**

12. How can you tell whether a given isometry is a reflection? How can you determine the location of the mirror? **Provide step-by-step instructions**.

13. How can you tell whether a given isometry is a translation? How can you determine the direction and length of the translation? **Provide step-by-step instructions.**

Special tools

1. Start up *GSP* and construct a triangle on the screen.
2. Construct the perpendicular bisectors of two of the sides of the triangle. Recall that the perpendicular bisectors of all three sides intersect at a single point. Construct this point of intersection (it is known as the *circumcenter* of the triangle).
3. Use the text tool (the "upper case A") to show the label of the circumcenter you just constructed.
4. Now, using the selection tool, double click on the label of the point. You will be given the option to change the name of the chosen point. Let's call it what it is by typing **`circumcenter`** then click **OK** (note that there are other options we could have chosen - e.g. we could have changed the font or type size, too).
5. What should be true about the distances from each vertex of your triangle to the circumcenter? Check to see whether *Sketchpad* agrees with your conjecture by measuring the distances from the triangle vertices to its circumcenter.
6. Now, construct the *circumcircle* of your triangle. It's defined to be the *unique* circle which contains the vertices of your triangle (this also helps to answer *why* we call the intersection of the perpendicular bisectors of the triangle's sides the *circumcenter*).
7. Take a few minutes to discuss with your neighbors why the circumcircle is, in fact, unique. That is, how do we know that there is **exactly one** circle which contains the vertices of your triangle?
8. If your circumcircle construction was done correctly, you should now be able to move your triangle around and the circumcircle should "follow". Be sure that this is the case before moving on.
9. **Hide** all of the extraneous stuff so that only the original triangle and the circumcenter are showing. Remember, to **Hide** objects you need to highlight them (don't forget that the **Shift** key allows you to highlight multiple objects) then choose **Hide** from the **Edit** menu (or type ****-h**).
10. Now, choose **Select All** from the **Edit** pulldown menu (or type ****-a**). All of your work should be highlighted.
11. Press and hold the **Custom** tool icon and choose **Create New Tool** from the Custom Tools menu that appears.
12. Type "Circumcircle" to name this new tool and click OK.
13. Press and hold on the Custom tool icon and select Circumcircle from the menu.
14. On an open part of the sketch, click on three points. What happens?

Activity:

Construct a triangle and its *circumcenter*. Next, construct its *orthocenter* (the intersection of the altitudes of the triangle - the altitudes are the lines through the vertices which are perpendicular to the opposite side) of the triangle and the *centroid* (the intersection of the segments connecting vertices to the midpoint of the opposite side). Label the *orthocenter, centroid,* and *circumcenter* and hide everything except these three points and the original triangle. Move the triangle around and make a conjecture about the relationship between the three points.

Graphing functions with *Geometer's Sketchpad*

In this activity, you'll learn to sketch functions and see how to use sketchpad to explore the effect of changing the coefficients in the graphs of equations like $y = ax + b$, $y = ax^2 + bx + c$, $y = ax^3 + bx^2 + cx + d$, and $y = A\sin(Bx + C) + D$.

1. Open a new sketch and choose "Show Grid" from the **Graph** menu.
2. Choose "New Function" from the **Graph** menu and type in `3x-5` and click **OK**. (Notice that $f(x) = 3x - 5$ appears on the screen.)
3. Being sure that the function is highlighted, choose "Plot Function" from the **Graph** menu.
4. There's another way to plot graphs of functions with *GSP*. Choose "Plot New Function" from the **Graph** menu and enter `3x+2` and click **OK**. See what happened?
5. Now, have *GSP* sketch the graphs of $y = 3x + 1$, $y = 3x$, $y = 3x – 1$, and $y = 3x – 3$.

This is a fairly common strategy to get students to see that lines with the same slope are parallel and to understand the effect of the slope and *y*-intercept on the graph of the line. But we can make use of some of the features of *GSP* to do make this more of an exploration.

6. Choose "New Parameter" from the **Graph** menu and name the parameter "a" by typing "a" and clicking **OK**.
7. Repeat this process to create parameters named "b", "c", and "d".
8. Choose "Plot New Function" from the **Graph** menu.
9. Click on the parameter "a" (on the sketch), then type "`*x +`" and then click on the parameter "b" (the entry should look like `a*x+b`). When you click **OK**, the function $ax + b$ should now be highlighted on the screen and the graph of $y = ax + b$ (for the stated parameters a and b, which are probably both equal to 1) will be sketched on the screen.
10. Highlight the parameter a and hit the "+" key on number pad at the far right of the keyboard. What happened? What happens if you hit the "-" key?
11. Right-click on a parameter and choose "properties" from the menu that appears. IN the "Parameter" submenu, change the "Keyboard (+/-) Adjustments" to 0.1 and observe what happens when you use the + and – keys.
12. Experiment with functions of the type $y = ax^2 + bx + c$, $y = ax^3 + bx^2 + cx + d$, and $y = A\sin(Bx + C) + D$, exploring the effect of changing parameters together or separately.