More lies my calculator told me:
Creating learning opportunities through calculator “mistakes”

Note: All examples apply to the TI-82 and 83, unless otherwise mentioned. There are analogous “lies” for the TI-73, 85, 86, and 92 calculators, when features are shared. An appendix of functions used is listed at the back of this handout.

Some graphing lies (for the TI-81, TI-82, and TI-83):

- We’re going to graph the function $y = \sin(28x)$. First, set your WINDOW to

```
Xmin=-10
Xmax=10
Xscl=1
Ymin=-2
Ymax=2
Yscl=1
```

Note: *This is just the* ZStd ZOOM *window with the* Y *range decreased (for better viewing).* **Be sure you’re in RADIANS mode!**

Now enter the function Y1=sin(28X) in the Y= menu. Those of you with a TI-85 or TI-86 can let y1=sin(41x) and if you have a TI-92, use **`sin(301x).`** Graph the function in the window specified above by selecting GRAPH now.

What do you notice about the graph of our function? Is it what you expected? What are some of the aspects of the graph that you didn’t expect? It might help to jot down a “description” (how would you describe the sketch over the phone if you didn’t have access to the function’s “formula”?). In particular, what’s the y-intercept and slope of the graph at the intercept?

- Still using Y1=sin(28X), let’s change our viewing window (just for the heck of it) to

```
Xmin=-5
Xmax=15
Xscl=1
Ymin=-2
Ymax=2
Yscl=1
```

At first glance, everything looks the same, but something strange is going on here! Look at the y-intercept again. What do you see now? Let’s try some other viewing windows:

```
Xmin=-8
Xmax=12
Xscl=1
Ymin=-2
Ymax=2
Yscl=1
```

```
Xmin=-6
Xmax=14
Xscl=1
Ymin=-2
Ymax=2
Yscl=1
```

- ***What’s going on?!! For a good laugh, try some other windows, keeping the “width” of the window 20. Now, imagine a classroom of kids each looking at the graph through a different window!***
- To show that it’s not just the window of “width” 20 for which this happens, look at the graphs of Y2=sin(50X) and Y3=sin(49X) using the ZTrig ZOOM (they *look* fine, at first, but there’s something **seriously** wrong with the graphs) *then* the ZInteger ZOOM (after changing Ymin to -2 and Ymax to 2 in order to see better).
- This forces us to ignore the calculator for a moment and consider what the graph *should* look like. What should the period of the function be? What’s the effect on the *actual* graph when you replace x with $28x$ (or $50x$ or $49x$) in the sine function?
- Finally, for a real eye opener, plot Y4=sin(18.8πX) in the following windows (then TRACE):

Xmin=-10
Xmax=10
Xscl=1
Ymin=-2
Ymax=2
Yscl=1

Xmin=-9
Xmax=11
Xscl=1
Ymin=-2
Ymax=2
Yscl=1

Xmin=-8
Xmax=12
Xscl=1
Ymin=-2
Ymax=2
Yscl=1

Note that ZOOMing out doesn't help! Can you see why? Spotting these "mistakes" requires a <u>nontrivial</u> understanding of the behavior of periodic functions (and the way a calculator plots functions). In fact, the calculator is not lying at all. If you TRACE ***along the curves above, the*** X ***and*** Y ***values that the calculator provides are approximately correct. All the calculator does is plot points and "connect the dots"; it's just that the points it plots are a function of*** Xmin ***and*** Xmax***. It is my hope that these "lies" convince at least <u>some</u> of my students to THINK before blindly trusting their calculator.***

<u>Determining limits by graphing:</u>

Now, let's look at another application of graphing - namely the use of graphs to determine (or make guesses about) limits and asymptotic behavior of functions. For example, it can be convincing, though not rigorous, to graph $y = \sin(x)/x$ in order to show that y approaches 1 as x approaches 0. But how do we know our graph is accurate?

- Enter Y5=sin(10000πX)/(10000πX) and graph the function using the ZStd ZOOM. Now TRACE near 0, and it certainly appears that the limit of this function as x approaches 0 is 0. But we should look at the graph "closer" to 0, right? OK, so ZOOM in and TRACE again. We get the same information, so let's ZOOM in again, just to be sure. Now some of you will be thinking "Of course we can't see anything; the limit is 1 and we've ZOOMED in so far that we can no longer see y=1." Good point, so let's change the WINDOW so that Ymin and Ymax are -2 and 2, respectively. While we're at it, we may as well set Xmin=-0.1 and Xmax=0.1. It's not until we have Xmin=-.01 and Xmax=.01 that it even appears that 0 *might not* be the limit. Of course, if we ZOOM in further, we'll finally get the impression that the limit *might* be 1, but after our previous experiences, how can we be **sure**?
- For another example in the same vein, change Y5 to sin(10000000πX)/(10000000πX), then graph and ZOOM as above. You'll see that you have to look even closer before you "see" the correct limit. **So, how do we know when we've looked "close enough". Of course, the answer is that you're never close enough to be sure. We need to use <u>mathematics</u> to determine the limit; not fancy *point plotting*.**

The next example shows that, when trying to compute the asymptotic behavior of a function (the limit of the function as x approaches ∞ or $-\infty$), we have similar difficulties.

- Enter Y6=x^2(e^(-(x/100)^2))*(1+e^((x-650)/4)) in the Y= menu, set the ZOOM to ZStd and graph Y6. Even when we trace the function, we might be convinced that we're looking at a parabola (of course we know that it's *not* a parabola). ZOOMing out twice doesn't really change that, but when you ZOOM out a third time, you see that this is *definitely* not parabolic. (At this point, you might want to change Xscl and Yscl to 0 in order to eliminate the "tick marks" on the axes). We're now looking at the WINDOW Xmin=-640, Xmax=640, Ymin=-640, Ymax=640 and it appears that the curve has a horizontal asymptote.
- Are you convinced that our function has a horizontal asymptote at y=0? Remember, we were almost convinced that the graph was parabolic a few ZOOMs ago! We could ZOOM out again, but let's manually change the viewing window, instead so that we Xmin=0, Xmax=1000, Ymin=0, Ymax=5000 since we're mainly interested in what happens as x approaches ∞. It certainly looks like we've got a horizontal asymptote, but just in case, we'd better TRACE.

  Oops! *What happens as x gets closer to 1000? Don't tell me it it's increasing now!*

- Let's increase the width of our window by setting Xmax=1100, then 1200.
  Oh great, it does blow up! Or does it? We've jumped to conclusions before; let's be more careful here.
- Set Xmax=2000 and we throw our hands up in disgust! Maybe a more algebraic approach might be called for, huh?
  The moral of the story is *you never know what might be behind the next ZOOM!*

More limits (this time using TABLE):

- In order to estimate $\lim_{x\to 0}\frac{\sin(x)}{x}$ we can graph $y = \sin(x)/x$ and TRACE toward the origin. But we can also define Y7=sin(10^(-X))/(10^(-X)) and set the TBLSET menu as below:

```
TABLE SETUP
  TblMin=1
  ΔTbl=1
Indpnt: Auto  Ask
Depend: Auto  Ask
```

By now looking at the TABLE, we can observe the values sin(.1)/.1, sin(.01)/.01, sin(.001)/.001, . . . corresponding to X=1,2,3,.... This is a nice use of the TABLE feature to gain some insight into the limit of our function as x approaches 0.

- Use this TABLE method to work with $\lim_{x\to 0}\frac{\sin(10000000\pi x)}{x}$. That is, change Y7 to equal sin(10^7π*10^(-X))/(10^(-X)), then bring up the table and make a guess about the value of the limit. **Oops! I was expecting 10000000π, or something close to it, but that's not what the calculator says (unless I avoid panicking and head further down the TABLE)! To see why this happened, think first about the values of X you're "plugging in" to sin(10000000πX) at the top of the table.**
- Now, use our method to investigate $\lim_{x\to 0}\sin(\frac{\pi}{x})$. That is, define Y8 to be sin(π/(10^(-X)) and look at the TABLE. ***Oops again! Do you suppose this is why our students don't think 0 is a number? We told them this limit doesn't exist!***
- We might as well keep going. Define Y9 to be (1+(1/(10^X)))^(10^X) in order to get an idea of the value of $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. It looks like the values are heading to *e* until the values become too big for the calculator to handle. ***Of course, in this case we can prove the limit is e, but how do we know this problem doesn't happen in other (less obvious) situations? Maybe there is some value in learning some techniques for doing limits "by hand".***

Now, you might say that some of these examples are artificial since every calculator or computer has a limit (pardon the pun) to the size of the numbers it can deal with (both big and small) and the results are just approximations, anyway. The following example, however, was an eye-opener for me.

"Repealing" the Fundamental Theorem of Arithmetic:

Not too long ago, I was teaching a Number Theory unit and had encouraged my students to use their calculators in order to find the remainder when dividing one integer by another. I'm sure that many of you are already familiar with this technique, but I'll illustrate with an example:

In order to determine the remainder when 11111111 is divided by 7, first have the calculator "compute" 11111111÷7. Since we get 1587301.571, we know that the quotient is 1587301, so the remainder is 11111111 - 1587301(7) = 4.

Now, on to the "lie". I gave my class an assignment to determine the remainders one gets when dividing powers of 5 by various integers. Here's what happened:

- Enter the number 5^14 and press ENTER (on a TI-85 or TI-86, enter 5^17). Now, press ÷4 (alternatively, enter 5^(14)÷4. Now, wait just a minute, here! How can I possibly get an integer result when I divide a power of 5 by 4? The last time I checked, the Fundamental Theorem of Arithmetic was still true! What's going on?!
- Subtract the "apparent" value of the answer from the answer given to us by the calculator (for the TI-83 or 82, type in "Ans - 1525878906") and we see that the calculator doesn't really "think" that 5^{14} is divisible by 4; apparently, it "remembers" more digits than it "admits". You might have already noticed that if you increase the power by 1 that the power of 5 will be expressed using scientific notation. In fact, this "lie" can be replicated on any calculator (that I've tried) by using the highest power of 5 that is *not* expressed using scientific notation.

I think this might be my favorite "lie" of all. For one reason, a student brought it to my attention (it's always nice to find they're *thinking* about what the calculator *tells* them). Another reason is that I now have an answer for students who ask "Why do we need to learn this divisibility stuff? Can't we just use a calculator?"

Occasionally, students also say, "OK, I guess I see your point about the calculator just being a very good approximator, but if all I'm interested in is an approximation, I can trust it, can't I?". Well, let's not be too hasty …

Approximating derivatives with the TI-82 or 83:

- Use the calculator to compute nDeriv(sin(1000πX),X,0), which *should* be an approximation of the derivative of $\sin(1000\pi x)$ at $x=0$. Of course, the actual value of the derivative should be $1000\pi \approx 3141.59$. ***Our approximation is not a very good one, since*** nDeriv ***uses the symmetric difference quotient - with $\Delta x = .001$ - to approximate the derivative.***

I should note that the TI-85 and 86 has a VERY good derivative approximator, der1, which provides an excellent approximation for every function available to the calculator, except for those functions involving the *greatest integer function* (or similar functions), in which case der1 returns an ERROR message.

Using sums to approximate integrals:

We know that if we want to approximate the integral $\int_0^1 \sin(x^2)dx$, we can use Riemann sums. For instance, our integral is approximately L_{10}, the left sum approximation with 10 subintervals, which equals

$$\sin(0^2)\cdot(\tfrac{1}{10}) + \sin((\tfrac{1}{10})^2)\cdot(\tfrac{1}{10}) + \sin((\tfrac{2}{10})^2)\cdot(\tfrac{1}{10}) + \sin((\tfrac{3}{10})^2)\cdot(\tfrac{1}{10}) + \cdots + \sin((\tfrac{9}{10})^2)\cdot(\tfrac{1}{10})$$

Of course, this can be represented using sigma notation as $\sum_{k=0}^{9}\sin((\frac{k}{n})^2)\cdot\left(\frac{1}{n}\right)$. However, I've found that the difficulty of learning sigma notation can often get in the way of understanding integral approximation, so I got the bright idea to make use of the **sum(seq(** command(s) in a way that does not require an understanding of sigma notation. For instance, without getting bogged down in details of sigma notation, we can compute sum(seq(sin(K^2)*(1/10),K,0,1-(1/10),1/10)) which equals the sum of all terms of the form sin(K^2)*(1/10) where K "starts" at 0 and stops at 1-1/10=9/10, increasing in increments of 1/10. You'll quickly recognize that this is, in fact, the LEFT SUM approximation of the integral using 10 subintervals. I found this to be a more efficient method which was less fraught with notational difficulties than sigma notation. All was going well until one day when my top student came in and said something must be wrong with his calculator since it was giving him answers that didn't make sense. Normally, I relish these moments, but this was different, since I had **no clue** as to what the problem was! He had made use of the table feature of the TI-82 to list a sequence of left sum approximations to the integral - namely $\boldsymbol{L_{10}}$,

L_{20}, L_{30}, ... by entering Y0=sum(seq(sin(K^2)*(1/(10X)),K,0,1-(1/(10X)),1/(10X))) which *really is* supposed to equal L_{10}, L_{20}, L_{30}, ... when X is chosen to be 1,2,3,..., respectively.

- Enter the above function into the Y= menu and set the TBLSET menu as before (TblMin=1). Now bring up the TABLE. We have the left sum approximations L_{10}, L_{20}, L_{30}, L_{40}, L_{50}, L_{60}, **and** L_{70} corresponding to X = 1,2,3,4,5,6, and 7 (well at least the calculator's estimates of them!). Our table should look like the one below:

X	Y6	
1	.2691	
2	.28946	
3	.29634	
4	.29981	
5	.30189	
6	.28956	

What "mistake" did the student notice in the entry corresponding to X=6? He knew that $f(x) = \sin(x^2)$ is an increasing function on the interval [0,1], so the sequence of left sum approximations should be increasing as well. But L_{60} is less than L_{50} in the table, so there's clearly a problem. In fact, L_{60} **should be** approximately 0.30328.

The problem arose because of the way the* sum(seq *program works. The reason, in a nutshell is as follows: When* X *is 6, rather than adding all terms of the form* sin(K^2)*(1/(60)) *where* K *ranges from 0 to 59/60 in increments of 1/(60), the sum continues as long as* K *is less than or equal to 59/60. Unfortunately, the calculator "skips" the 60th piece of the sum because of roundoff error (*K*, according to the calculator, is slightly larger than 59/60, so it thinks you want to "stop"). The "good" thing about this mistake is that the students now see the necessity of sigma notation (or a good integration approximation program!).

- If we redefine Y9 to be sum(seq(sin((K/(10X))^2)*(1/(10X)),K,0,10X-1,1) This should be the *honest to goodness* left sum approximations L_{10}, L_{20}, L_{30}, L_{40}, L_{50}, L_{60}, **and** L_{70} we were trying to get before. After selecting both Y0 and Y9 for the TABLE, notice how often the two answers differ. I wonder how many mistakes we made in class by believing the calculator?

Fun with fractions:

There's a useful key (**F↔D**) on a TI-73, which "translates" between fractional and decimal notation. Just for fun, type in .3333333333, then the **F↔D** key, then **ENTER**. *Not exactly what you expected, is it?*

Final comments:

The reaction I get from students is interesting. Many of them become paranoid and worry that I'm going to constantly give them problems on which I know the calculator is going to lie. I often have to reassure them that I won't always try to trick them with a "lie". Several of them do, however, "get it" - realizing that the calculator is only an approximator which must be used intelligently. When I use these examples in class, my hope is that my students will come to a greater appreciation of the calculator as a useful learning tool but I also hope that they understand that *it can't "do mathematics"*, at least by itself. And most importantly, if we're going to be able to recognize these calculator lies, we're going to have to *know some mathematics*, ourselves.

I hope these examples have been illuminating. I also hope I've made it clear that I'm an enthusiastic proponent of the *appropriate* use of technology in the classroom, but I think an important aspect of appropriateness (yes, it's really a word) is an understanding of the limitations of the technology being used. These "lies" at least get the conversation started. I'm always on the lookout for more, so please email with any you run across.

Appendix (a list of functions used in this presentation and a few other "lies"):

1. Y1=sin(28X)
   (Xmin=-10, Xmax=10, Ymin=-2, Ymax=2) (Xmin=-8, Xmax=12, Ymin=-2, Ymax=2)
   (Xmin=-5, Xmax=15, Ymin=-2, Ymax=2) (Xmin=-6, Xmax=14, Ymin=-2, Ymax=2)
   [use sin(41x)for TI-85 and 86; sin(301x) for TI-92 and DERIVE]
2. Y2=sin(50X), Y3=sin(49X) (using ZTrig then ZInteger)
3. Y4=sin(18.8πX) (use WINDOWs as in #1; Are sine curves horizontal?)
4. Y5=sin(18000πX)/(10000πX) (using graph, conjecture limit as x approaches 0)
5. $y = x^2 e^{-(x/100)^2}(1 + e^{\frac{x-650}{4}})$ Y6=X^2*e^(-(X/100)^2)*(1+e^((X-650)/4)))
   What is $\lim_{x \to \infty} y$? See graph (you never know what's behind the next ZOOM!).
6. Y7=sin(10^(-X))/(10^(-X)) (for $\lim_{x \to 0} \frac{\sin(x)}{x}$ using TABLEs)
7. Then, change Y7 to sin(10^7π*(10^(-X)))/(10^(-X)) (for $\lim_{x \to 0} \frac{\sin(10000000\pi x)}{x}$ using TABLEs)
8. Y8=sin(π/(10^(-X)) (using TABLEs, you get a surprise "answer" for $\lim_{x \to 0} \sin(\frac{\pi}{x})$)
9. Y9=(1+(1/(10^X)))^(10^X) (for $\lim_{x \to \infty}(1 + \frac{1}{x})^x$ using TABLEs)
10. 5^(14) /4 (on the 85 and 86, use 5^(17) /4)
    (you can also get "interesting" results by dividing by 3, 2, and 6)
11. nDer(sin(1000πX),0) (of course der1 on the TI-85 or 86 works much better)
12. Y0=sum(seq(sin(K^2)*(1/(10X)),K,0,1-(1/(10X)),1/(10X)))
    an alternative to sigma notation (or a program) for Left sum approximations of $\int_0^1 \sin(x^2)\,dx$ (with 10, 20, 30, etc. subintervals) but L_{60} is wrong.
13. (on a TI-73) .3333333333 **F↔D ENTER**
14. Don't forget the old standbys: 1/2X vs. 1/(2X)
    the TI-83 "fixed" the problem of the TI-82: sin2X, sin2*X, sin(2X)
15. An old mistake on the TI-85 (correct on the 82, 83, 86): Compute the determinant of the "generic" 3X3 matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. It SHOULD be 0.