1, 4, 9, What's next? *42, of course!*

1. 1, 4, 9, ______ Explain how you arrived at your solution.

2. What is the 100^{th} term in the above sequence? Explain.

There seem to be two different methods used when solving problem 1, both of which result in the solution 16. The choice of method greatly affects the relative difficulty of problem 2, even though both result in an answer of 100.

The two methods:

Recognize a formula: Noticing that $1 = 1^2$, $4 = 2^2$, and $9 = 3^2$, the next term is $4^2 = 16$ and the 100^{th} term is $100^2 = 10{,}000$

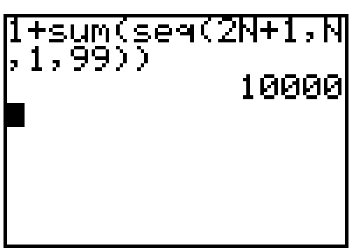
Look at successive differences: 4 – 1 = 3 (and odd number) and 9 – 4 = 5 (the next biggest odd number), so 4^{th} term minus 9 should equal 7, the next odd after 5). But the 100^{th} term is a little more difficult (or at least time consuming) to solve for. Notice that the first difference is 2(1)+1, the second difference is 2(2)+1, and the third difference is (we think) 2(3)+1, so the 99^{th} difference (which, when added to the 99^{th} term in the sequence will be the 100^{th} term) will be 2(99)+1, or 199. Then the 100^{th} term of the sequence should be 1 plus the first 99 differences:

$1 + (3 + 5 + 7 + \ldots + 199) = 10{,}000$ (not surprisingly)

Note: To compute this with the TI-83 or TI-84, type

1 + LIST (2^{nd} STAT) **MATH 5** (to get sum) **LIST OPS 5** (to get seq)
2 alpha LOG + 1 , alpha LOG , 1 , 99) ENTER

This should give you the following result:



In general, to mimic sigma notation, you need to use the sum(seq(combination. The sum function adds up the elements of a list and the seq (short for sequence) function creates a sequence (or list) determined by an algebraic formula. For example, to compute the sum of the cubes of the third through twenty-eighth counting number, evaluate **1+sum(seq(N^3,N,3,28)** (the lone "N" is the "variable" of the sequence).

Did anyone come up with alternative answers to these questions? Who's to say the sequence continues to increase? Maybe the 4^{th} term is 1 because the sequence cycles on 1, 4, and 9: 1, 4, 9. 1. 4. 9. … . Or perhaps the next term is 22 (13 more than 9 – do you see why a student might choose that one?) The moral of this story is that there aren't enough terms to establish a pattern, even if we believe there is a pattern to be found.

Here's another problem to try:

3. What's the next term of the sequence 3, 6, 11, … ? What's the 100^{th} term? Explain.

This problem is a lot more difficult to solve with the *recognize a formula* strategy, but it's the same as the first problem if you use the *successive differences* method, since the successive differences are *exactly the same* as for the sequence 1, 4, 9, … .

What if the pattern isn't "obvious" (or worse, doesn't exist)?

4. Find the next term in the sequence: 9, 4, 1, 8, 27, __

The most common response is 64 (do you see how someone might make that guess?), but it's wrong.

The answer is in the paragraph after this one. Trust me, you're probably not going to guess the answer – nobody ever has (who hasn't known the trick). But if you insist on thinking about it, don't read below this line until you give up.

This excerpt from http://www.bostonredsox.com gives away the answer.
1931: The Red Sox players first wear numbers on their uniforms. Since then, the Red Sox have retired five uniform numbers: Ted Williams' No. 9 and Joe Cronin's No. 4 officially retired May 29, 1984; Bobby Doerr's No. 1 retired May 21, 1988; Carl Yastrzemski's No. 8 retired August 6, 1989; and Carlton Fisk's No. 27 retired September 4, 2000. Major League Baseball retired the No. 42 of Jackie Robinson.

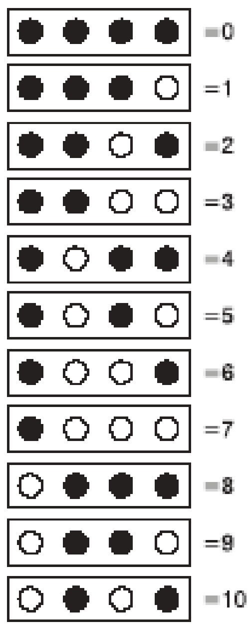
"But wait a second", you say, "this problem isn't fair. I have to know how you picked the numbers in order to correctly guess the next term in the sequence."

Exactly!

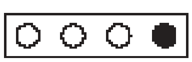
I would go even further and point out that, on an all-or-nothing high-stakes test, all "what's the next term" problems are inherently unfair, since they require the student to read the mind of the test writer. By no means do I think "guess the next term" problems are bad classroom problems. They provide excellent opportunities for students to flex their problem solving muscles as well as giving them a chance to explain their solution method.

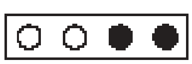
For the last straw, see the next two pages, showing an example from the 2002 grade 10 MCAS. While the story had a happy ending, think about what would have happened had the young woman not questioned AUTHORITY and spoken out about what she thought was a correct solution?

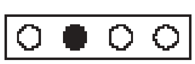
8 Computers are designed around off/on switches that are used to represent numbers. In the following pattern, which represents the numbers from 0 to 10, ○ represents a switch that is on and ● represents a switch that is off.

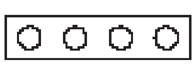


Which of the following represents the number 11?

A. 

B. 

C. 

D. 

Reporting Category for Item 8: ***Patterns, Relations, and Algebra***

From the Massachusetts Department of Education press release (December 4, 2002)

Points Awarded To Students Who Selected Alternate Answer On Grade 10 MCAS Exam

MALDEN - An additional 449 students in the classes of 2003 and 2004 have earned a competency determination, thanks to the ingenuity of a Whitman-Hanson Regional High School student who found a unique method of answering a math question on the 10th grade exam, education officials announced on Wednesday.

Because of this finding, a second answer on the question is now counted as correct. As a result, an additional 136 students in the class of 2003 and 421 students in the class of 2004 have now passed the math exam, and will not have to take the MCAS math retest being given next week.

“Although the answer we had marked as the right one is correct, it’s clear now that what this student found is also right, and I think that’s terrific,” said Education Commissioner David P. Driscoll. “This girl was able to take a typical math question and come up with a completely unique method of solving it that even our math experts, teachers in the field and our test reviewers never considered. This is a great example of just how creative our students can be, and I applaud her efforts.”

The question presented a real-world application of the binary, or base two, number system, such as the one used by computers. The numbers zero through 10 were shown as a sequence of on and off switches in four-switch panels. Students were asked to identify the panel that would represent number 11, the next in the sequence. The correct answer could be found either by knowing the binary system, or by recognizing changes in place value throughout the series.

However, when the student looked at the panels, she saw something different: a spatial sequence in the on and off switches, rather than the numeric pattern envisioned by the developers and reviewers of the item. After conferring with mathematicians, education officials this week determined that the student’s answer is also a viable second solution to the question.

Back to the method of successive differences

Consider the sequence given by the formula $f(x) = x^2$ (that is, the first output is $1^2 = 1$, the 2^{nd} output is 2^2=4, the third output is 9, etc.

The successive differences for the terms of the this sequence is a sequence itself:

	1		4		9		16		25		
1st differences Δ :		3		5		7		9		11	
2nd differences Δ^2:			2		2		2		2		
3rd differences Δ^3:				0		0		0			

5. Find the successive differences, Δ, Δ^2, Δ^3, and Δ^4 for the sequences defined by each of the following formulas

a. $f(n) = 3n + 4$

b. $p(n) = 5n - 7$

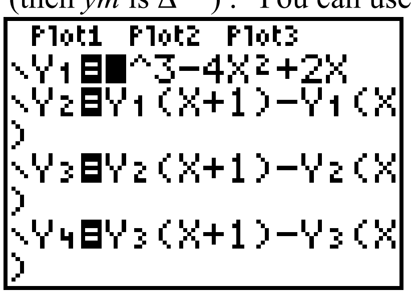
c. $g(n) = n^2 + 3n – 4$

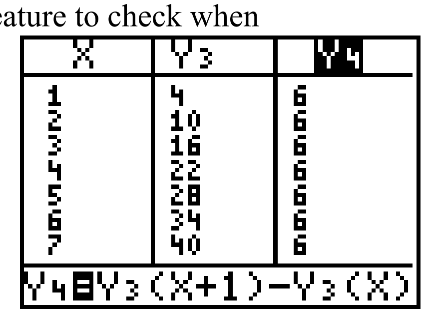
d. $h(n) = 3n^2 - 5n + 1$

e. $j(n) = 2n^3 + n$

f. $k(n) = n^3 - 4n^2 + 2n$

Calculator hint:
If $y1$ is the sequence, then define $y2(x) = y1(x+1)-y1(x) = \Delta^1$, $y3(x) = y2(x+1) - y2(x) = \Delta^2$, etc. (then ym is Δ^{m-1}) . You can use the table feature to check when





Using the TI-89 (as in the figure on the right), you can expand these functions symbolically. If you're only interested in whether a function is constant, check a few specific values

6. What do you notice about the successive differences? Do the differences ever become constant? When?

7. Replace the question mark and fill in the blank:

- If the sequence has a polynomial formula of degree n, then $\Delta^{?}$ is constant.

- If the formula for the sequence is $a_n x^n + a_{n-1}x_{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, then $\Delta^n =$ ____.

Let's investigate a little further, shall we?

It's not too hard to show by hand that if $f(x) = ax+b$, then $\Delta^1 = a(x+1)+b-(ax+b) = a$.

If $f(x) = ax^2+bx+c$, then $\Delta^1 = a(x+1)^2 + b(x+1) + c - (ax^2 + bx + c) = 2ax + a + b$ and $\Delta^2 = \Delta^1(x+1) - \Delta^1(x) = 2a$

What if the polynomial has degree 3?

8. Enter $y1(x) = ax^3 + bx^2 + cx + d$, $y2(x) = y1(x+1) - y1(x)$, $y3(x) = y2(x+1) - y2(x)$, $y4(x) = y3(x+1) - y3(x)$, and $y5(x) = y4(x+1) - y4(x)$. (Then $y1$, $y2$, $y3$, $y4$, and $y5$ represent Δ^1, Δ^2, Δ^3, and Δ^4, respectively. Next, enter expand(y4(x)) on the Home screen. Record your answer.

9. Change $y1$ to $ax^4 + bx^3 + cx^2 + dx + e$ and record the result of expansion of $y5$ (representing Δ^4).

Do the results of problems 8 and 9 confirm your conjectures from problem 7? It's actually not too difficult to prove, using the principle of mathematical induction, that if the sequence has a polynomial formula of degree n, then $\Delta^n = n!$ times the leading coefficient of the polynomial. This proof is left to the reader, unless there's time in the workshop to pursue it.

You're probably mumbling to yourself, "The title of this session is *1, 4, 9 What's next? 42, or course!*" Are you going to explain that? Of course, the *real* reason is that 42 is the answer to the ultimate question of Life, the Universe, and Everything (according to the late Douglas Adams), but if you insist, here's what you're really looking for:

So, how do you get 42 as the 4th term of the sequence 1, 4, 9, …?

The goal for the remainder of the workshop is to find a formula for the sequence 1, 4, 9, … that will justify the choice of 42 as the 4th term. To begin, let's just find a polynomial that gives us the correct first three terms (pretending for a moment that we don't already know one).

If we were to start from scratch, how could we build a function, f, so that $f(1) = 1$, $f(2) = 4$, and $f(3) = 9$? We could start by trying $f(x) = 1$. That works for $x = 1$, but not for $x = 2$ or 3. Similarly, I could define $f(x) = 4$ or 9. I'm going to do something a little strange and define f as follows:

$$f(x) = \quad 1 \quad + \quad 4 \quad + \quad 9$$

Of course, this is a silly definition, since it doesn't give the right output for *any* of the inputs 1, 2, or 3. However, 1 is what we want when x=1, 4 is what we want when x=2, and 9 is what we want when x=3. Let's call 1 the "1-part" of the above formula, call 4 the "2-part" and call 9 the "3-part". Is there a way to "fix" the 1-part so that it will be 0 when $x = 2$ and 3? What's a function that's equal to 0 when x=2? When x=3? Think about this before continuing. Then check to see if the function below satisfies the property we wanted.

$$f(x)= \mathbf{1}(x-2)(x-3) + \mathbf{4}(x-1)(x-3) + \mathbf{9}(x-1)(x-2)$$

Notice that the 1-part will be 0 when x=2 or 3, the 2-part is 0 when x is 1 or 3, and the 3-part is 0 when x is 1 or 2. Therefore, the only input affecting the 1-part is 1, the only input affecting the 2-part is 2, and the only input affecting the 3- part is 3. Unfortunately, we still don't get the correct outputs! When x=1, we get an output of 1(1-2)(1-3) = 2 (not 1), when x=2, we get 4(2-1)(2-3) = -4 (not 4), and when x=3, we get 9(3-1)(3-2) = 18 (not 9). How can we fix it without changing the fact that each part is 0 for the correct inputs? Again, think about this before continuing.

$$f(x)= \frac{\mathbf{1}(x-2)(x-3)}{(1-2)(1-3)} + \frac{\mathbf{4}(x-1)(x-3)}{(2-1)(2-3)} + \frac{\mathbf{9}(x-1)(x-2)}{(3-1)(3-2)}$$

$$= \tfrac{1}{2}(x-2)(x-3) - 4(x-1)(x-3) + \tfrac{9}{2}(x-1)(x-2)$$

Using the expand function on the TI-89 (or another Computer Algebra System), we can simplify this polynomial (of course, we can easily do this by hand, but use the tools you've got, if you've got them, that is).

Did it surprise you that we got x^2? After all, it's a quadratic polynomial.

So, how do we find a function satisfying $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, *and* $f(4) = 42$?

Here are a few. Do you see how they were derived? Are they all the same? Can you think of any others?

$$x^2 + 26\frac{(x-1)(x-2)(x-3)}{6} = x^2 + 26\frac{(x-1)(x-2)(x-3)}{(3-1)(3-2)(3-3)}$$

$$x^2 + 26\frac{x(x-1)(x-2)(x-3)}{24}$$

$$\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 4\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} + 9\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 42\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$