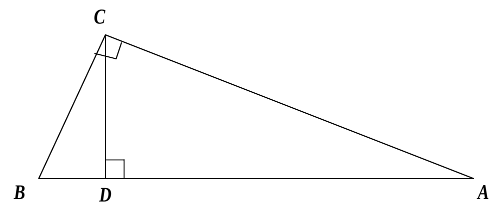
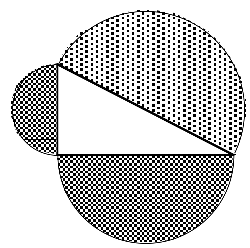
4. Find the perimeter of ΔABC, shown below, given that $CD = 12$, $BD = 9$, and angles ADC and ACB are right angles.



5. In the figure below, the legs and hypotenuse of a right triangle are the diameters of semicircles. How does the sum of the areas of the smaller semicircles compare to the area of the larger semicircle?



6. If you know that $\sin(\theta) = -\frac{5}{13}$, what can you say about the value $\cos(\theta)$?

2. Pythagoras' *first* cousins

In the What is Mathematical Investigation *module, you play these games with the consecutive sums problem.*

In mathematics, it is often very fruitful to play the "What if?" and "What if not?" games with the hypotheses and context of a known result. Let's take the Pythagorean Theorem as an example (now there's a shocker!).

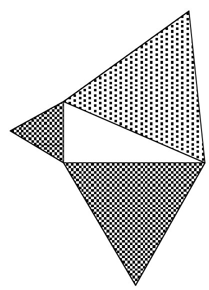
The Pythagorean theorem was originally stated, by Euclid for instance, in terms of areas:

> *The area of the square on the hypotenuse is the sum of the areas of the squares on the legs.*

In fact, mathematicians of ancient Greece did not separate numbers from geometry—products, like squares, had to denote areas *What if*, instead of building squares off the sides of a right triangle, you built some other sort of figure? As you showed in the first session of this module, if you build semicircles off the sides, the area of the semicircle built on the hypotenuse is the sum of the areas of the semicircles built on the two legs. You subsequently showed in the *Further Exploration* materials of session 1 that the same result is true if the semicircles are replaced by squares or equilateral triangles, but *not* true if the semicircles are replaced by rectangles with constant height. In case you didn't work on those problems earlier, take a moment to do so, now.

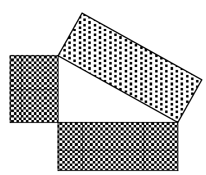
1. Construct equilateral triangles on the sides of a right triangle, as in the figure below. Is the area of the triangle on the hypotenuse equal to the sum of the areas of the triangles on the legs?

What if *the squares are replaced by equilateral triangles?*



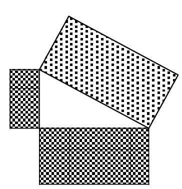
2. In the figure below, rectangles of equal height have been built off the sides of a right triangle. Is the sum of the areas of the smaller rectangles equal to the area of the larger rectangle?

What if *the squares are replaced by rectangles of equal height?*



3. Construct rectangles on a right triangle so that the base of each rectangle is a side of the triangle and the height of each rectangle is half the base. Is the area of the rectangle on the hypotenuse equal to the sum of the areas of the rectangles on the legs?

What if *the squares are replaced by rectangles half as tall as they are wide?*



4. Among the last few attempts at generalizing the Pythagorean theorem by replacing squares with other figures, only problem 2 didn't "work." What about the hypotheses of that problem were different from those in problems 1 and 3? Make a conjecture, which is as general as possible, specifying the types of figures that can be placed on the sides of a right triangle so that the area of the figure on the hypotenuse is guaranteed to be equal to the sum of the areas of the figures on the legs.

What other Pythagorean cousins have you thought of? In Session 3, you will meet some more by exploring the question, "*What if* $\triangle ABC$ isn't a right triangle? How does the sum of the areas of squares built of the legs compare to the area of the square built off the hypotenuse?"

3. Pythagoras' nontriangular cousins

Although the Pythagorean Theorem is about triangles—specifically, right triangles—it has applications to other objects, in both two and three dimensions. And, as you probably guessed, the theorem has several non-triangular cousins, as well.

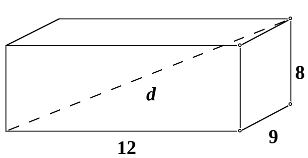
Of course, "the square of the diagonal" is shorthand for "the square of length of the diagonal."

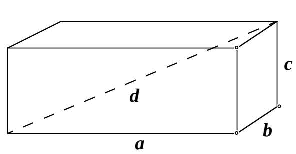
1. Find a relationship between the sum of the squares of the diagonals and the sum of the squares of the four sides of a rectangle.

Surprise!

2. Is there a relationship between the sum of the squares of the diagonals of a parallelogram and the sum of the squares of the sides?

So far, we've only delved into 2-dimensional Pythagorean cousins. Does the Pythagorean theorem have any 3-dimensional generalizations or applications? Let's see!





3. A rectangular box is 12 inches long, 9 inches wide, and 8 inches deep. What's the furthest distance apart two points on the box can be from one another?

4. What's the relationship between the three dimensions of a rectangular box (length, width, and height) and the length of its "diagonal"?

5. What's the distance between the points $(1, 2, 3)$ and $(0, 4, 5)$ in space?

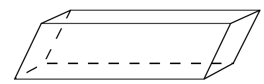
6. Find an equation for the sphere of radius 4 centered at the point $(1, -1, 2)$.

You saw in problem 2 that the sum of the squares of the four sides of a parallelogram is the sum of the squares of the two diagonals. Alternatively, one could say that the sum of the squares of the two different edge lengths is the *average* of the squares of the diagonals. Problem 4 shows that the same can be said of the relationship between the edges of a rectangular box and its diagonals.

Note that each edge and diagonal length appears four times in the figure.

7. Is there a relationship between the sum of the squares of the edges and the average of the squares of the diagonals in a "parallelepiped" (a slanting box, in which opposite sides are congruent parallelograms)?

A parallelepiped is the three-dimensional analogue to a $2-D$ parallelogram.

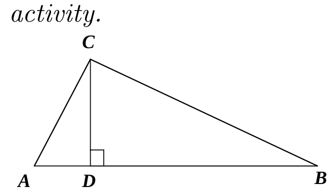


4. Constructing *nice* problems

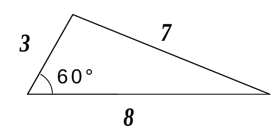
If you have taught Geometry or Trigonometry, the next problems are probably similar to some you've given in class. ∢ Take a few minutes to solve them in order to be aware of what's involved when students work on them. Later, we'll discuss the *creation* of these problems.

You might choose to provide your students with more information, however, depending on the goal of the activity.

1. Determine the perimeter of triangle ABC in the figure below, given that $AC = 13$, $CD = 12$, and $BD = 16$. *What if* $AC = 15$ (while CD is still 12 and BD is still 16)?



2. Compute the area of the triangle having side lengths 3, 7, and 8, given that one of its angles measures 60 degrees. *What if* the side lengths were 7, 13, and 15 (and one of the angles was still 60 degrees)?



3. Determine the lengths of the sides of the triangle having vertices at the points $(1, 2)$, $(10, 14)$, and $(5, 2)$.

4. A rectangular box is 12 inches long, 9 inches wide, and 8 inches deep. What's the furthest distance apart two points on the box can be from one another?

This is problem 10 *from session* 3. *Solve it again to remind yourself of its features.*

Did you notice that all of the numbers in these problems were integers? That wasn't by accident. While there are certainly cases where "messy" numbers are appropriate, there are times when it's preferable for the numbers in the question and solution to "come out nice." In the article "Meta-problems in mathematics," Al Cuoco wrote,

Read more about it: *"Meta-problems in mathematics," appeared in the November* 2000 *issue of the* College Mathematics Journal, *published by the Mathematical Association of America.*

> *I have a conjecture: A great deal of classical mathematics was invented by teachers who wanted to make up problems that come out nice. Problems that come out nice allow students to concentrate on important ideas rather than messy calculations. They give students feedback that they are on the right track. They are easier to correct.*

But how do we construct problems that come out nice but aren't the ones the students have already seen? Some of the questions you will address in this session (corresponding to the creation of problems 1–4 above) are:

- How can we find right triangles with integer-valued side lengths?
- How can we find triangles with side lengths and (at least) one altitude which are integer-valued?
- How can we find triangles with integer-valued side lengths with vertices at integer-valued coordinates?
- How can we find rectangular boxes with integer-valued side lengths *and* diagonal?

Pythagorean triples

Most of the problems on the previous page involved right triangles with integer side-lengths. Triples of the form (a, b, c), where a, b, and c are positive integers satisfying the equation $a^2 + b^2 = c^2$, are called *Pythagorean triples.* Some fairly well-known Pythagorean triples are(3,4,5), (and its *cousins* (6,8,10), (9,12,15), ...), (5,12,13), and (7,24,25). While we're often happy for students to recognize Pythagorean triple patterns, there might also be occasions in which we *want* them to do the required arithmetic to solve for one of the values in the equation $a^2 + b^2 = c^2$.

Right triangles with integer-valued side lengths are called ***Pythagorean triangles****.*

Can you think of any others? If not, that's OK; we're going to derive a formula which gives us ***all*** *of them.*

So, how can you find more Pythagorean triples that aren't just multiples of ones you already know? One way is to use trial-and-error: pick two integers for a and b and see if $\sqrt{a^2 + b^2}$ is an integer. Alternatively, you could pick integer values for a and c and check whether $\sqrt{c^2 - a^2}$ is an integer. Are there any ways to narrow our search? Are there properties Pythagorean triples must satisfy that we can apply?

The triple (ka, kb, kc) is said to be a multiple of the triple (a, b, c).

The table below lists several related Pythagorean triples. Do you see how they are related? Can you find the next few triples in the table?

a	b	c
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
13	84	85

5. Find a pattern in the table, then determine the next 3 triples by following the pattern.

Check that the triples you add to the table are Pythagorean triples.

15		
17		
19		

6. Use the pattern you found in problem 5 to create a formula that will generate infinitely many Pythagorean triples. Prove that your formula always gives a Pythagorean triple.

7. Here's another table of Pythagorean triples. Find a pattern, then fill in the next 3 triples in the table by following the pattern.

Be sure that the triples you create are Pythagorean triples.

a	b	c
4	3	5
8	15	17
12	35	37
16	63	65
20	99	101
24		
28		
32		

8. Based on the previous table, guess another formula that will generate infinitely many Pythagorean triples. Prove that your formula always gives a Pythagorean triple.

9. Here's another list of Pythagorean triples, which includes each of the triples already given plus a few more. Use it to compile a list of at least 3 conjectures you think might be true of *all* Pythagorean triples.

For now, you don't need to prove that your conjectures are correct, but be sure that the triples in the table satisfy your conjectures.

a	b	c
3	4	5
5	12	13
6	12	15
7	24	25
8	15	17
9	40	41
10	24	26
11	60	61
12	35	37

a	b	c
13	84	85
14	48	50
15	112	113
16	63	65
17	144	145
18	24	30
19	180	181
20	99	101
21	220	221

Some More Pythagorean Triples

The last few problems provided some ways of finding infinitely many Pythagorean triples, but there are still more triples that can't be found with either of the formulas you discovered nor are they multiples of any of the triples you get by these formulas. There is a way of finding every single Pythagorean triple, though. In *Further Exploration*, you'll have the opportunity to work through the derivation of the following formula, which generates *all* Pythagorean triples.

A Pythagorean triple formula
If n and d are positive integers, then $(\lvert n^2 - d^2\rvert, 2nd, n^2 + d^2)$ is a Pythagorean triple. In fact, every triple of this form is a Pythagorean triple *and* every Pythagorean triple is a multiple of a triple of this form.

In other words, every Pythagorean triple looks like $(k(n^2 - d^2), 2knd, k(n^2 + d^2))$ for some integers n, d, and k such that $n > d$.

Before continuing, take a moment to ponder the above statement. The formula provides a way to find *every* Pythagorean triple there is. That's amazing!

A practical benefit of the formula is that it can be used to write a fairly simple program for a computer or programmable calculator to list as many triples as you want.

10. Show that if n, d, and k are positive integers and $n > d$, then $(k(n^2 - d^2), 2knd, k(n^2 + d^2))$ is a Pythagorean triple.

11. Use the formula to find five Pythagorean triples that don't appear in the table on page 8.

12. Find all Pythagorean triangles with at least one side of length 12. ▷

A Pythagorean triangle is a right triangle with integer-valued side lengths.