

Promoting Mathematical Habits in High School: Some Snapshots

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... the future well-being of our nation and people depends not just on how well we educate our children generally, but on how well we educate them in mathematics and science specifically.

—*John Glenn, September, 2000*

The only people who need to study calculus are people who want to be calculus teachers.

—*Bill Cosby, April, 1999*

Mathematics constitutes one of the most ancient and noble intellectual traditions of humanity. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought as well as the concepts and language for precise quantitative description of the world around us. *It affords knowledge and reasoning of extraordinary subtlety and beauty, even at the most elementary levels.*

—RAND Mathematics Study Panel, 2002

Implications for Curriculum Design

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—*the habits of mind*—used to create the results.

How Does This Play Out in Students' Experience?

Let's look at two mathematical habits:

- reasoning by continuity (from analysis)
- abstracting regularity from repeated calculations (from algebra)

Examples Taken From "The CME Project"

a comprehensive four-year NSF-funded high school curriculum forthcoming from Prentice Hall

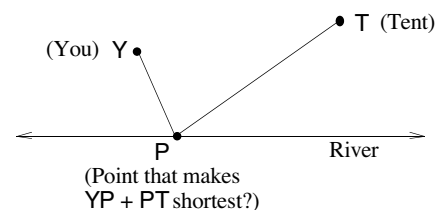


with additional support from Texas Instruments



Two of the Writers

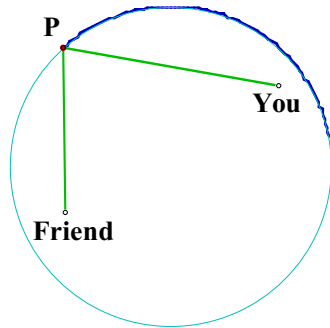
Reasoning by Continuity



Where to Land?

You are on a camping trip, returning from a walk. Standing at A , you notice that your tent (located at B) is on fire. You need to run to the river, fill your bucket with water, and get to the tent. Where should you land on the shore to minimize the total distance of the trip?

You are in a circular swimming pool at A , you want to swim to the edge of the pool to drop off your wine glass, and then swim to your friend at B . Where should you land on the edge of the pool to keep the trip to a minimum?



Now What?

Abstracting Regularity

Find a function that matches each table:

Input	Output
0	3
1	8
2	13
3	18
4	23

Input	Output
0	1
1	2
2	5
3	10
4	17

Table A: Two Models

- $f(n) = 5n + 3$

-

$$g(n) = \begin{cases} 3 & \text{if } n = 0 \\ g(n-1) + 5 & \text{if } n > 0 \end{cases}$$

Question:

$$f \stackrel{?}{=} g$$

$$g(n) = \begin{cases} 3 & \text{if } n = 0 \\ g(n-1) + 5 & \text{if } n > 0 \end{cases} \quad f(n) = 5n + 3$$

Two Possible Directions

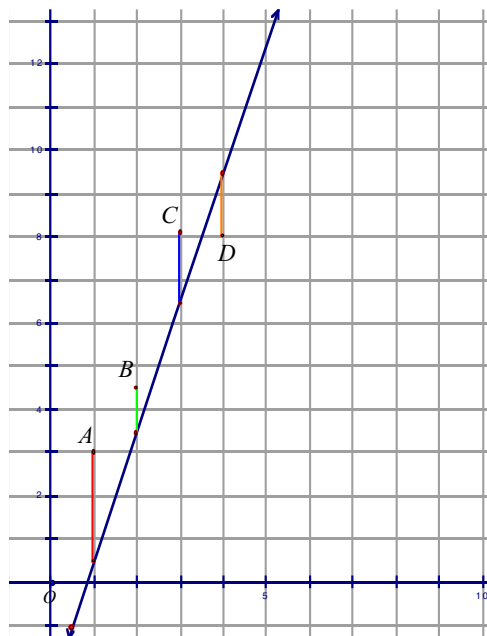
- Regression Lines
- Monthly Payments

Preferences?

Regression Lines

Here is some made-up data:

Input	Output
1	3
2	4.5
3	8.1
4	8



The Errors

Here's a useful image to build in your head: Imagine that you have a "moveable" line on the scatter plot, one that you can control (with a mouse, say). As the line moves, some sticks grow in length, while others shrink. If you keep track of the sum of the squares of the lengths of the sticks, you could fine tune the line and adjust it to make the badness small. You can turn this thought experiment into a real experiment if you use your geometry software. Here are the steps: . . .

Of all lines with slope 3, which is “best?”

Data vs. Line Fit: $y = 3x + b$

Input	Output	Predicted:	Error:
1	3	$3 \cdot 1 + b = 3 + b$	$3 - (3 \cdot 1 + b) = -b$
2	4.5	$3 \cdot 2 + b = 6 + b$	$4.5 - (3 \cdot 2 + b) = -1.5 - b$
3	8.1	$3 \cdot 3 + b = 9 + b$	$8.1 - (3 \cdot 3 + b) = -.9 - b$
4	8	$3 \cdot 4 + b = 12 + b$	$8 - (3 \cdot 4 + b) = -4 - b$

So, we want to minimize

$$(-b)^2 + (-1.5 - b)^2 + (-.9 - b)^2 + (-4 - b)^2$$

Ah ... This is a *quadratic* in b . And we know how to minimize a quadratic

This simplifies (by hand or CAS) to:

$$4b^2 + 12.8b + 19.06$$

So, the minimum value is when

$$b = \frac{-12.8}{2 \cdot 4} = -1.6$$

So, of all lines with slope 3, the best one has equation

$$y = 3x - 1.6$$

Play the same game with different slopes, and you find a certain rhythm to the calculations.

Slope	Badness	Minimizing value of b
0	$158.86 - 47.2b + 4b^2$	5.9
1	$52.26 - 27.2b + 4b^2$	3.4
2	$5.66 - 7.2b + 4b^2$.9
3	$19.06 + 12.8b + 4b^2$	-1.6
⋮	⋮	⋮

... And a surprise ...

Slope	Equation of best line	y -intercept	Δ
0	$y = 5.9$	5.9	-2.5
1	$y = x + 3.4$	3.6	-2.5
2	$y = 2x + .9$.9	-2.5
3	$y = 3x - 1.6$	-1.6	-2.5
4	$y = 4x - 4.1$	-4.1	

The y -intercept seems to depend *linearly* on the slope:

$$b = 5.9 - 2.5m$$

What does this say geometrically?



Various best lines

From Here ...

- The y -intercept seems to depend linearly on the slope:

$$b = 5.9 - 2.5m$$

- The point of concurrency seems to be $(2.5, 5.9)$
- And $(2.5, 5.9)$ is the *centroid* of the data

All this can be established via a careful analysis of algebraic calculations that ramp up to full generality. And then,

- Students develop an algorithm for finding the best line (of *any* slope), and
- This algorithm is encapsulated into a *formula* for the line of best fit.

An Algebraic Interlude

In PROMYS each summer, students investigate the following question:

Given a positive integer n , how many integer factors does it have?

For example, how can I tell how many factors 72 has without actually listing them and counting?

Now, let's investigate a related algebraic questions. Start, of course, with a numerical experiment:

n	Number of Irreducible Factors of $x^n - 1$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Theorem: If n is a positive integer, we have a formal identity:

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

This is sometimes called the *cyclotomic identity*.

If n is prime, the second factor is irreducible over \mathbb{Z} (Eisenstein).

If n is composite, the second factor splits some more—
Its factorization is quite wonderful and mysterious.

Monthly Payments

Suppose you want to buy a car that costs \$10,000. You don't have much money, but you can put \$1000 down and pay \$350 per month. The interest rate is 5%, and the dealer wants the loan paid off in three years. What kind of car can you buy?

This leads to the question

“How does a bank figure out the monthly payment on a loan?”

or

“How does a bank figure out the balance you owe at the end of the month?”

Take 1

What you owe at the end of the month is what you owed at the start of the month minus your monthly payment.

$$b(n, m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n - 1, m) - m & \text{if } n > 0 \end{cases}$$

Take 2

What you owe at the end of the month is what you owed at the start of the month, *plus* $\frac{1}{12}$ of the yearly interest on that amount, minus your monthly payment.

$$b(n, m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n - 1, m) + \frac{.05}{12}b(n - 1, m) - m & \text{if } n > 0 \end{cases}$$

Students can then use successive approximation to make

$$b(36, ???) = 0$$

Except ...

$$\begin{array}{c}
 b(4) = b(3) + \frac{.05}{12} \cdot b(3) - 250 \\
 \swarrow \quad \searrow \\
 b(2) + \frac{.05}{12} \cdot b(2) - 250 \quad b(2) + \frac{.05}{12} \cdot b(2) - 250 \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 \hline
 \end{array}$$

It takes too much %!*&\$ work

Take 3

Same as Take 2, with a little algebraic simplification for efficiency:

What you owe at the end of the month is what you owed at the start of the month, *plus* $\frac{1}{12}$ of the yearly interest on that amount, minus your monthly payment.

$$b(n, m) = \begin{cases} 9000 & \text{if } n = 0 \\ \left(1 + \frac{.05}{12}\right) b(n-1, m) - m & \text{if } n > 0 \end{cases}$$

Students then use successive approximation to make

$$b(36, ???) = 0$$

Let's try it:

**What's the monthly payment
on a loan of \$9000 for 36 months?**

Pick an interest rate and keep it constant. Suppose you want to pay off a car in 24 months. Investigate how the monthly payment changes with the cost of the car:

(1) Make a table like this:

Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
⋮	⋮

(2) Describe a pattern in the table. Use this pattern to find either a closed form or a recursive rule that lets you calculate the monthly payment in terms of the cost of the car in thousands of dollars. Model your function with your CAS and use the model to find the monthly payment on a \$26000 car. Check your result with the original "b" program

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x	y(x)	Δ
0	-3	>30
1	23.7	>30
2	67.7	>30
3	111.7	>30
4	155.7	>30
5	199.7	>30
6	243.7	>30
7	287.7	>30
8	331.7	>30
9	375.7	>30
10	419.7	>30
11	463.7	>30
12	507.7	>30
13	551.7	>30
14	595.7	>30
15	639.7	>30
16	683.7	>30
17	727.7	>30
18	771.7	>30

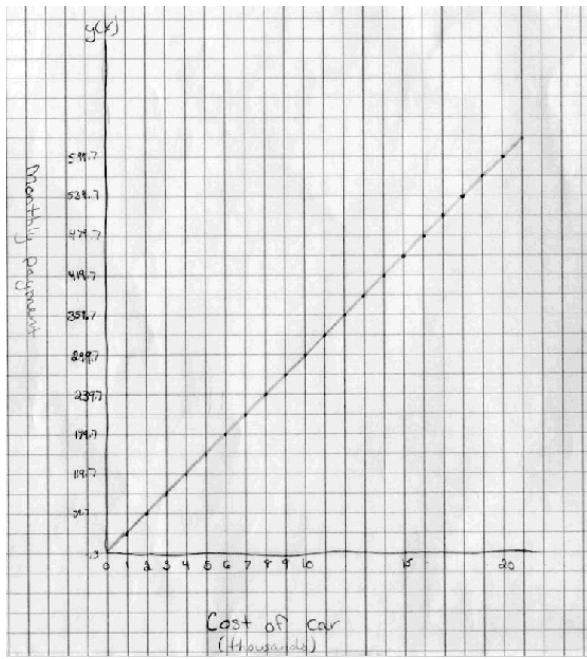
b) $y(x) = \begin{cases} -3 & \text{if } x=0 \\ (x-1)+30 & \text{if } x>0 \end{cases}$ ✓ describe pattern at (or base)

$y(x) = x+30?$

d) \$86000 car \$779.7 monthly payment ←

c) $y(x)$
func
.P x=0 Then
Return -3
Else
Return $(x-1)+30$ ✓

- I changed the amount of the cost of the car then I changed the monthly payment until I found the right monthly payment.
- I found that each time the cost of the car went up \$1000 the monthly payment went up \$30.



The balance at the end of 36 months with a monthly payment of \$250 can be gotten by entering $b(36, 250)$ in the calculator:

: $b(36, 250)$
: 764.92

:

Because we are using a CAS, we can do it *generically*: The balance at the end of 36 months with a monthly payment of m can be gotten by entering $b(36, m)$ in the calculator:

: $b(36,m)$

: $10453.3 - 38.7533 \cdot m$

So,

: $\text{solve}(10453.3 - 38.7533 \cdot m = 0, m)$

: $m = 269.738$

A monthly payment of \$269.74 will do the trick.

Suppose you borrow \$12000 at 5% interest. Then you are experimenting with this function:

$$b(n, m) = \begin{cases} 12000 & \text{if } n = 0 \\ (1 + \frac{.05}{12}) \cdot b(n-1, m) - m & \text{if } n > 0 \end{cases}$$

Notice that

$$1 + \frac{.05}{12} = \frac{12.05}{12}$$

Call this number q . So, the function now looks like:

$$b(n, m) = \begin{cases} 12000 & \text{if } n = 0 \\ q \cdot b(n-1, m) - m & \text{if } n > 0 \end{cases}$$

where q is a constant.

Then at the end of n months, you could unstack the calculation as follows:

$$\begin{aligned} b(n, m) &= q \cdot b(n-1, m) - m \\ &= q(q \cdot b(n-2, m) - m) - m = q^2 \cdot b(n-2, m) - qm - m \\ &= q^2(q \cdot b(n-3, m) - m) - qm - m \\ &= q^3 \cdot b(n-3, m) - q^2m - qm - m \\ &\vdots \\ &= q^n \cdot b(0, m) - q^{n-1}m - q^{n-2}m - \dots - q^2m - qm - m \\ &= 12000 \cdot q^n - m(q^{n-1} + q^{n-2} + \dots + q^2 + q + 1) \end{aligned}$$

The last series is geometric; summing it, we get

$$b(n, m) = 12000 q^n - m \frac{q^n - 1}{q - 1}$$

Setting $b(n, m)$ equal to 0 gives an explicit relationship between m and the cost of the car:

$$m = 12000 \frac{(q-1)q^n}{q^n - 1}$$

or, in general,

$$\text{monthly payment} = \text{cost of car} \times \frac{(q-1)q^n}{q^n - 1}$$

where n is the term of the loan and

$$q = 1 + \frac{\text{interest rate}}{12}$$