An Optimization Module

Contents

The best way to understand something is to teach it to someone else.

— Anon.

What this is about

We'll say "secondary school" to mean middle and high school.

This is a mathematics book for people preparing to teach mathematics in middle and high school. In writing the book, we had four major purposes in mind:

Some say that ideas from calculus can be introduced in elementary school.

Sometimes, calculus is called the "mathematics of change."

- We want you to make connections between the mathematics you learn as an undergraduate and the mathematics you will teach in secondary school. In particular, there are many central ideas and themes from calculus and analysis that can be introduced to students quite early. That's not to say that most secondary students should learn to differentiate and integrate functions defined by complicated formulas. Learning to do that is only one aspect of learning calculus. What we're talking about is having secondary students work with the *ideas* that the techniques formalize.

  In particular, this book is concerned with the idea of *optimization*: given a system of some sort, what conditions bring it to a desirable state? In this book, you'll study several techniques for solving optimization problems that use high school geometry, elementary algebra, and some basic ideas about functions and graphs. We'll use the tools to foreshadow the much more powerful tools of calculus and analysis. Another important idea from calculus is the notion of *continuous change*. A formal study of continuity needs to wait for calculus, but secondary students can gain a head start by building intuitions about dynamically changing systems, especially if they use dynamic geometry software (like Cabri and Geometer's Sketchpad) as a medium to model the systems. We'll show you what we mean in the many experiments and investigations in this book that make basic use of dynamic geometry software.
- We want to help you gain a deeper understanding of the mathematics in the undergraduate curriculum and in the secondary curriculum. It really is true that a deeper understanding of a topic comes from trying to make it accessible to someone else. If you can get the essence of continuity across to a ninth grader, then you have learned a great deal about teaching *and* about continuity. If you can explain the

ideas behind maximizing a system without saying "take the derivative and set it equal to zero," then your understanding of maximization techniques has deepened significantly.

It's not just that we want you to be able to make advanced ideas tractable to students without advanced backgrounds; we also want you to see pre-college mathematics differently. The almost hackneyed problem of finding a rectangle of a given perimeter and maximum area is in all the eighth grade algebra books as an exercise in graphing quadratic functions and finding vertices of parabolas. But we want you to deepen your own understanding and to be able to communicate the "big picture" as you see how the problem connects to other mathematics: to inequalities, to geometric dissections, and to continuous variation.

Undergraduate mathematics courses need to prepare students for engineering, business, graduate school in mathematics, and other fields. There's not always time to connect the material to secondary mathematics. These connections are the primary focus of this book. The major context we use is that of a secondary school classroom.

- We want you to develop ways of thinking that are typically mathematical. We call these mathematical "habits of mind" and our experience is that the major disconnect between mathematics as a discipline and mathematics in secondary school is due to the fact that school curricula tend to emphasize the *results* of mathematical thinking rather than the thinking itself. For example, look at the table of contents of any school algebra text ("reformed" or otherwise). It contains things like solving equations and inequalities, simplifying expressions, graphing lines and curves—all very reasonable things for students to know. But often a mathematician or someone who uses mathematics in a job looks at such a book and says that it misses the point. The "point" is that there is an algebraic approach to things, a collection of algebraic habits of mind, that almost never get discussed in school texts. The result is that algebra becomes a collection of exercises that have little point for either the teacher or the student. It doesn't have to be that way.

  This book represents our attempt to let you (and hopefully your students) in on the process of doing mathematics. We'll discuss mathematical ways to think about the problems we pose, and ask you to reflect on your own thinking. There are in this book results and techniques, to be sure, but the real emphasis is on the mathematical habits of mind that lead to the results.

- In addition to all this, we want you to think about how students learn—and how teachers teach—these mathematical ideas. Part of this entails thinking about how *you* are learning new mathematical concepts and ideas as you learn them and discussing your insights with others. Another part of it

> is figuring out how to help students to ask the right questions, the questions they need to answer in thinking about a problem. We ask you to reflect upon various student dialogues, to think about students' thinking, to think about the contexts in which they learn, to focus on what they might gain from a particular problem or set of problems, and to think about how to invite students to articulate their ideas and to develop the level of inquiry in the classroom. All of that, in order to help *students* make mathematical connections, deepen their understanding, and develop mathematical habits of mind.

So, that's what we are trying to do: Make mathematical connections, deepen understanding, bring attention to mathematical thinking, and focus on the pedagogy. All this will develop in the context of working on problems that we think you could use in your teaching. In addition to all you'll learn, we think the problems are fun, and we hope you enjoy working on them as much as we did.

Introduction to Optimization

Much of a typical calculus class is devoted to using differentiation techniques to find absolute and relative maxima and minima for various functions. The ideas behind looking for maxima and minima—thinking about extreme cases and boundary conditions, and using a mixture of deduction, experimentation, and reasoning by continuity—are accessible to secondary students without the use of calculus. It just takes the right problems and the right approach.

This book presents several optimization problems common in secondary curricula that benefit from these approaches (as well as other problems). Working through this book, you will learn some new mathematics, connect some of the ideas from secondary mathematics to ideas from calculus, and think about what secondary students might gain from working on these problems.

The problems that follow ask you to start thinking about different optimization situations and techniques. As the book progresses, you'll start to develop some mathematical ways of thinking about problems like these (we call these ways of thinking "mathematical habits of mind"). For now, work on each problem using any method you think might help.

1. A food pantry receives a donation of 17 pounds of peanut butter all in one big tub, from the Good and Nutty company. It needs to pack the peanut butter into two size jars; the big one holds two pounds, and the small one holds 9.5 ounces. How many jars of each should be packed so that the minimum amount of peanut butter is left over?

A pound is 16 ounces.

Maximize what you pack into the jars and minimize the amount you have left over.

2. A friend visiting from London needs to change British pounds to American dollars. A currency exchange service will give you \$1.55 for each £1 with no service charge. The bank down the street will give you \$1.65 for each £1, but they'll charge you \$2 for the transaction. Where should you go to change the money?

Questions to Ask: *Does it matter how much money you have? What if you have £10? £300?*

3. Find different pairs of integers that sum to 20. Check the product of each pair. Of all these pairs which has the greatest product?

4. (a) Which rectangle of a fixed perimeter will enclose the most area?

(b) What if you are not limited to rectangles? In other words, what *shape* of a fixed perimeter encloses the

Questions to Ask: *If it seems difficult to think about a generic rectangle, what about a specific perimeter, like 21?*

most are?

Write and Reflect:

5. How are problems 3 and 4a related?

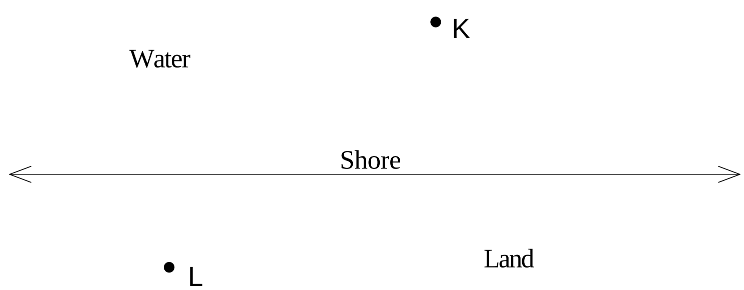
Part 1. Geometric Techniques

In this part, you will develop some ways to solve optimization problems using geometry. As you work, think about what ideas and results from secondary mathematics you use.

1. Minimizing Distance

The problems that follow focus on finding the shortest path between different locations.

1. You are lounging on the beach at L, and you want to run to the shore and swim out to your friends on a raft at K.



A run and a swim.

(a) You want the swim to be as short as possible. To what point on the shore should you run to minimize the path you swim? Why?

Make sketches to show where you should hit the shore line.

(b) You want to reach K with the least running. At what point on the shore should you enter the water now? Why?

(c) You want to reach K in the least possible total distance. *Now* where should you hit the shoreline? Why?

2. A pipeline needs to be built between two oil wells, and a pump will service both of them. The location of the pump should minimize the cost of new pipeline. Where should it be placed?

Write and Reflect:

3. Consider using problem 2 in a high school class. What problems might the context of the problem cause? How

One of the debates that goes on among problem posers—including teachers and the authors of textbooks—is the role and importance of context in mathematics problems.

would the students' answers and discussion be different if you used one of these contexts instead:

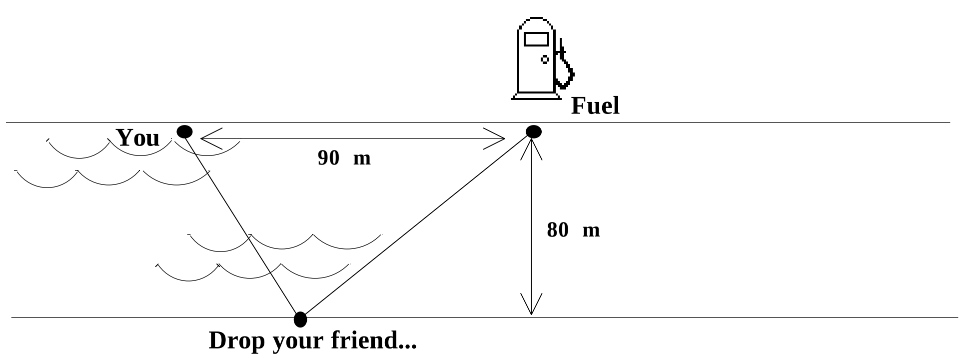
- Two cities decide to pool their resources to build a recreation center. Because the cities are going to build new roads, the location of the recreation center must minimize that cost. Where should the recreation center be placed?
- You have two points, A and B, and you want to find a point C so that the total distance $AC + BC$ is a minimum.

The simplest way to find the shortest distance between two points in the plane is to draw the segment that connects them. However, sometimes you don't take the shortest path from one point to another because there is *a third place* you want to visit first. Consider the following problem, common in many secondary curricula:

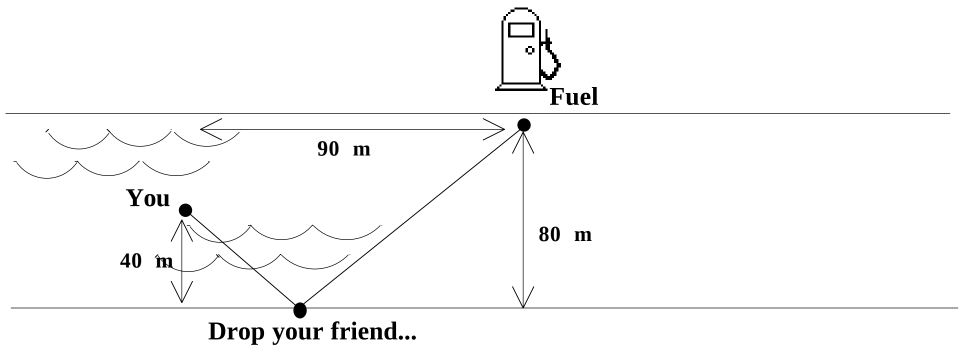
Modeling this situation in dynamic geometry software can be helpful.

4. Imagine that you're motorboating on a river, and the boat is very short on fuel. You *must* drop a passenger off on the south riverbank first, and then you can go to refuel at a station on the north riverbank. You are really short of fuel, so you need to minimize the total distance of the trip. Where should you drop off the passenger?

(a) First, suppose you are near the north bank, 80 meters from the south bank, and that the refueling station (also 80 meters from the south bank) is 90 meters downriver from you. How far along the south bank should you stop?



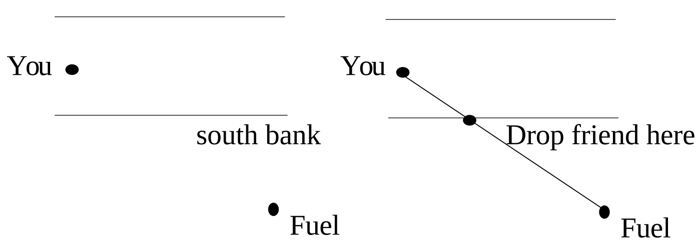
(b) Now, suppose you are near the middle of the river—40 meters from the south bank—and that refueling station is still 80 meters from the south bank and 90 meters downriver from you. How far along the south bank should you stop?



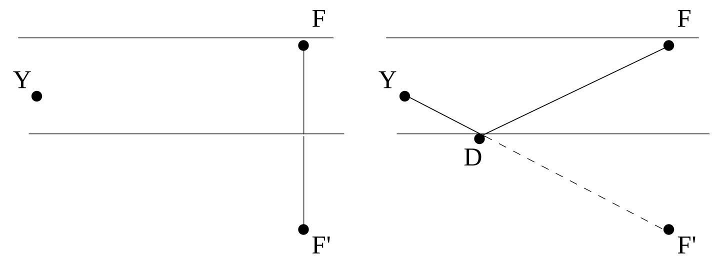
Thinking About Student Thinking:

Chris, a high school student, proposed this solution:

If the fuel station were on land, on the same side where I want to drop off my friend (and if boats could move on land), it would be easy: It's just the "swim and run" problem we did earlier.



The shortest trip from *You* to the *Fuel* would be along the segment $\overline{YF}$, so the best spot to drop your friend is where that segment crosses the south bank. So I reflected the F over the south bank to get F', then connected F' and Y. The place where it crosses the river is the best spot to land.



5. Explain Chris's technique:
   (a) How do you reflect a point over a line? (Like when the student reflected F over the shoreline.)
   (b) Compare the lengths of $\overline{FD}$ and $\overline{F'D}$. Does the same relationship hold for points on the river other than D? Why or why not?
   (c) Show that any other point on the river requires a longer total trip.

Chris used a very useful habit of mind in mathematics: reducing a problem to one you already know how to solve. Chris knew how to find the shortest distance between two points, and found a way to turn the total distance from point to a line to another point into a distance between two points. We'll call this technique *the reflection principle*.

Notice that the reflection principle provides a general solution to the problem, not just a solution for the two cases given. No matter where Y and the F are located, you can use the reflection principle to find the best spot to land.

Write and Reflect:

6. Restate the reflection principle in your own words, and explain why it works.

Thinking About Student Thinking:

Terry, another student in class, objected to Chris's technique:

That solution won't work in this case! You can't just put the fuel station on land. You're in a boat, and it can't go on land. The fuel station is in a fixed place and you can't move it. You can only move the point where you drop off your friend. You have to solve the problem the way the boat could do it.

For Discussion:

What do you think of Terry's complaint? How could you (as a teacher) respond to it?

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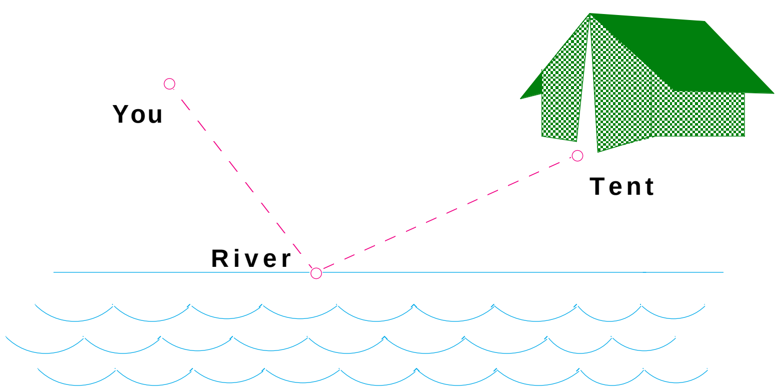
7. You're on a camping trip. While walking back from a hike, you see that your tent is on fire. Luckily you're holding a bucket and you're near a river. Where should you get the water along the river to minimize your total travel back to the tent? Justify your answer.

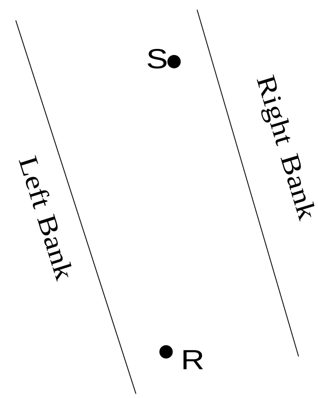
We'll see this problem about the "burning tent" again.



Where along the river should you stop?

8. In the picture, you are in a canoe in a river at R. You must first let a passenger out on the left bank, then pick up a passenger on the right bank and deliver that passenger to an island at S. Explain how to find the drop-off and pick-up points that minimize the total distance traveled.

Questions to Ask: *Can you use the same habit of mind Chris used, turning this into a problem about finding the shortest distance between two points? Would a reflection of* S *suffice? A reflection of* R*?*



Back and forth . . .

Hint*: Your sunglasses can be left at any one of the pool's four sides.*
We're not looking for the spot so much as the method *you use to find it*

9. You're in a rectangular swimming pool at K, out of reach of the sides of the pool. Before swimming to L, you want to swim to a side of the pool to put down your sunglasses. Explain how to find the place to put your sunglasses that minimizes the length of the path you swim.

Don't lose your sunglasses.

Write and Reflect:

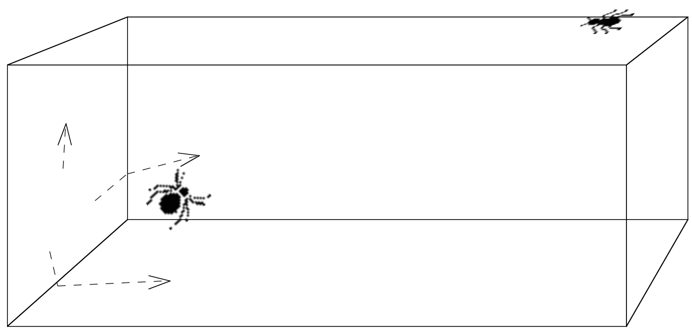
10. If your students had no clue about how to solve problems 7–9, what kinds of things could you do to help them get started?

11. What facts, results, and methods from high school geometry have you used in solving these problems?

Take it Further

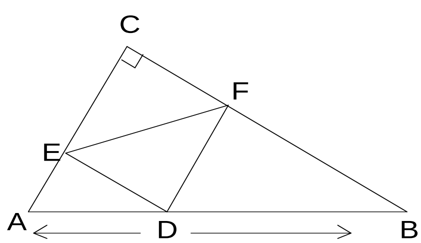
12. A hungry spider sees a bug it can eat on the ceiling of a room. The picture below shows the spider on the left side wall. What would be the shortest path along the walls between the two? (No air travel.) Make a sketch to show what you think is the shortest path, but make sure to *describe how* to find the shortest path as well.

Questions to Ask: *Can you reduce this to a problem about the shortest distance between two points on a plane?*



Which way?

13. In $\triangle ABC$, $\angle C$ is a right angle, D is a moving point on $\overline{AB}$, $\overline{DE} \perp \overline{AC}$, and $\overline{DF} \perp \overline{BC}$.



Where should D be along $\overline{AB}$ to minimize EF?

(a) Set up a sketch of this situation in your dynamic geometry software and find the best place for D experimentally.

(b) Find the *exact* best spot for D and prove your claim.

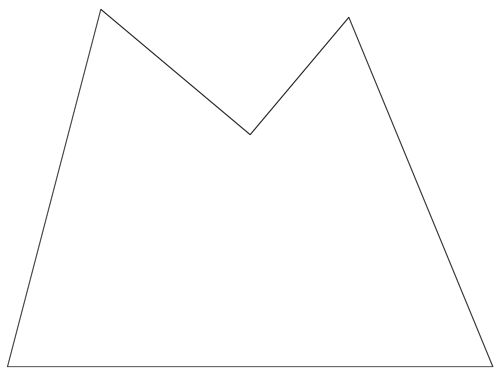
(c) Investigate the problem if $\angle C$ is not required to be a right angle.

Write and Reflect:

14. How would you respond to a high school student who asks about problem 13b, “What’s needed to prove my claim?”

2. Maximizing Area

Just as you sometimes want to minimize things, there are times you want to *maximize* things, to make them as large as possible. In general, people want to minimize things that cost money, require a lot of boring work, waste time, or make them uncomfortable. Perhaps the most common thing geometers look to maximize is area.



1. (a) Describe a way to make a polygon with larger area than this one, but with exactly the same length sides.
(b) How do you know you have a polygon of larger area?

Here is a problem common to many middle and high school curricula. Solve it any way you can.

2. The expense of building a house depends greatly on its perimeter; that is where the expensive things like windows and doors are found. Suppose you want to build a house with a rectangular base. Given your budget, you decide you can afford a house with total perimeter of 128 feet. What dimensions should you choose for the base of the house if you want to maximize its area?

Thinking About Student Thinking:

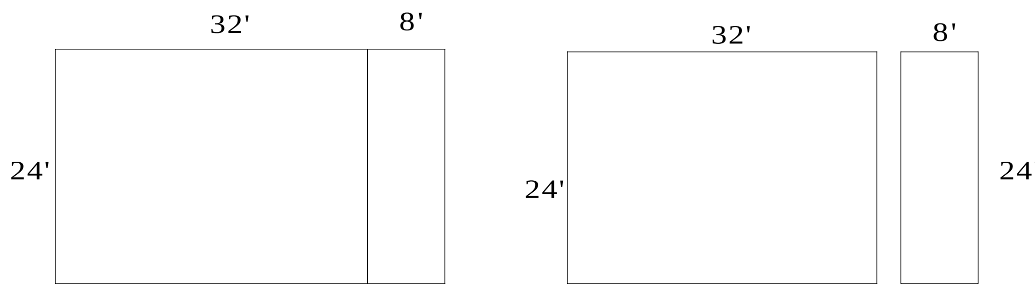
Jo came up with the following argument to show that a square with 32 feet on a side is the best solution to problem 2:

I'll show that a 32×32 square is best by demonstrating that any other rectangle with a perimeter of 128 has area smaller than the area of the square. I'll do this by showing that I can cut up such a rectangle and make it fit inside the 32×32 square with room to spare.

Is this a good example? What's the perimeter of a 40×24 *rectangle?*

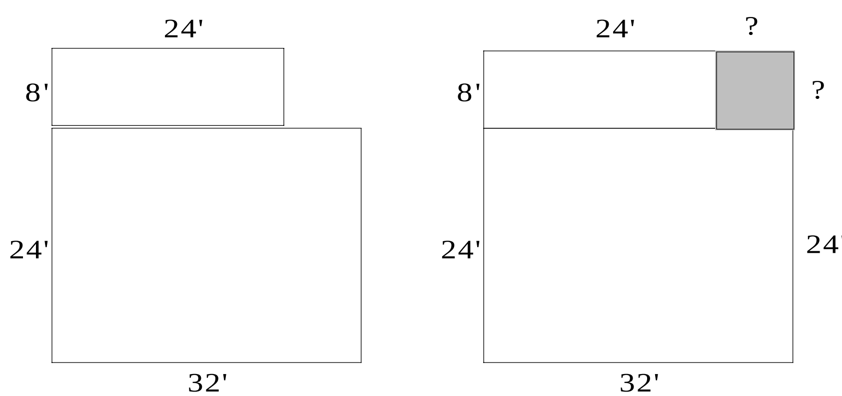
Suppose, for example, I have a 40×24 rectangle.

First, I'd cut it like this:



Snip. Then rearrange these ...

Then, I take the small strip off the side and put it on top of the 24×32 rectangle:



... to get this ... ... Which is not quite a square.

The operation *rearranged* the area, but I didn't add or lose any area. Meanwhile, my two pieces cover *some* of the 32×32 square, but not all of it. The shaded part isn't covered, so the square has more area. Therefore, the area of the 32×32 square is *bigger* than the area of the 40×24 rectangle.

3. This argument works for *this* rectangle, but does it generalize to *all* rectangles of perimeter 128? Rewrite the cutting argument for a general rectangle of perimeter 128 feet.

Questions to Ask: *If the length of a rectangle is l, and the rectangle's perimeter is 128, what is the width of the rectangle in terms of l?*

4. How big is the "uncovered" square in the student's cutting argument? How big is it in your argument for a generic rectangle with perimeter 128 feet?

5. Rewrite the cutting argument to show that an $a \times b$ rectangle has a smaller area than a square with the same perimeter. How big is the uncovered square in this case?

6. Problem 5 asks you to use a cutting argument. Consider the same problem algebraically and see where it leads you.

Questions to Ask: *What is the perimeter of an $l \times w$ rectangle? What is its area? What is the area of a square with the same perimeter?*

For Discussion:

Here are three different versions of this problem. The first we posed earlier, the second you just worked on, and the third is one example of other versions commonly found in curricula. How does the context affect the problem and the possible solutions students might find? Is the problem about rectangles worse than the ones about houses or rabbits? How might the addition of context cause problems in student understanding?

Another discussion about the role of context.

- Which rectangle of a fixed perimeter will enclose the most area?
- The expense of building a house depends greatly on its perimeter; that is where the expensive things like windows and doors are found. Suppose you want to build a house with a rectangular base. Given your budget, you decide you can afford a house with total perimeter of 128 feet. What dimensions should you choose for the base of the house if you want to maximize its area?
- You have 20 feet of fencing with which to build a rectangular rabbit pen. What shape pen would the rabbit like best?

Take it Further

7. What if you were not limited to building a rectangular house? If you could use any shape base for your house, what shape with perimeter 128 feet would enclose the most area?

8. Triangles of many different sizes may have two sides of length 5 and 6. Which of these triangles has the most area?

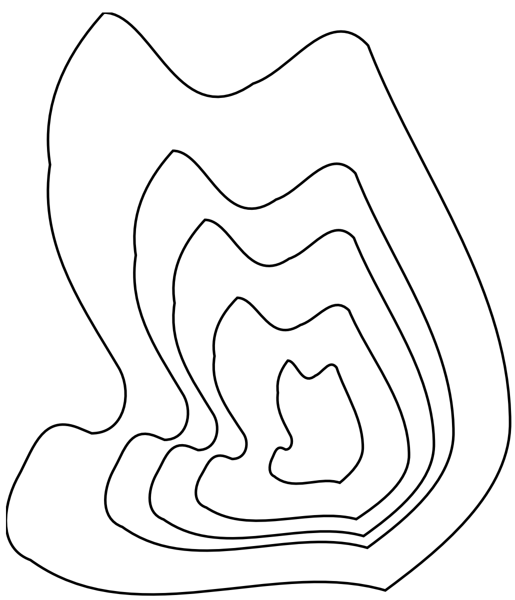
3. Contour Lines

There is a way of thinking about certain optimization problems that ties together many topics in precollege mathematics. Traditionally, this "method of contour lines" is reserved for advanced calculus, but recent developments in technology, (especially geometry software), and some old-fashioned gadgets made from pins and string make it possible to use this method long before calculus. The contour line method can help students get qualitative solutions to many optimization problems, and to build *ideas*. The formal machinery of calculus will later help students make these ideas more precise.

Introduction

A topographic map usually depicts land elevations and sea depths by curves or lines that represent points of about the same elevation:

All the points on the same curve are about the same distance above or below sea level. Is this a map of a mountain? A lake?

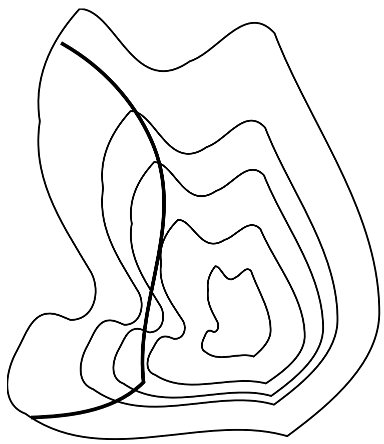


Although this diagram shows only closed curves, other maps may include open (partial) curves. It depends on the size and scale of the map.

1. Make a contour plot of a school gym (with all the bleachers pulled out), of a local sports stadium (include the stands), of an outdoor amphitheater (like *Blossom* in Ohio or *Great*

See if people in your class can guess the place described by your map.

Woods in Massachusetts), or of an auditorium or music hall that you've visited.



2. Suppose the map on page 19 is the topographic map for a mountain, and there's an increase of 1000 feet between contour lines.

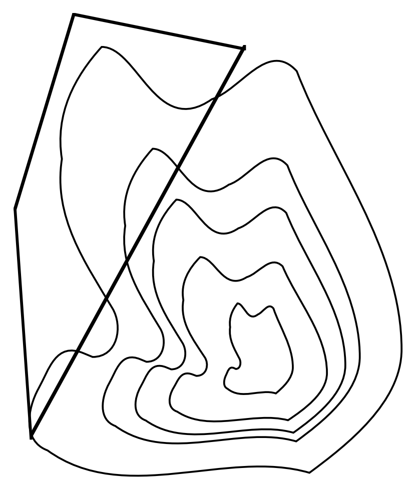
(a) If the lowest part of the mountain is 100 feet above sea level, sketch the places on the mountain that are about 1600 feet above sea level.

(b) The exact peak is unmarked here, but given what you know about these contour lines, what is the maximum elevation it could be?

(c) What is the steepest part of the mountain? Explain.

3. If the picture at the side shows the path of some hikers on the mountain, what is the highest elevation they reach on their hike?

4. This same picture could be a map of a pond, whose outer contour line represents the edge of the pond, with each inner contour line representing an increase in depth of 10 feet. Imagine a camp owns the land inside the quadrilateral shown in the figure here, and the swim director wants to rope off a children's swimming area with water no deeper than 5 feet. Trace the map on another piece of paper and sketch a proposal for such a roped-off area.

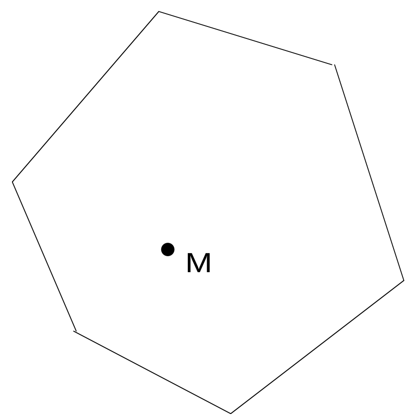


Words: *Why would people use the words* contour *and* level *to describe these paths?*

The individual curves in a contour plot are called *contour lines* or *level curves*. You can think of a contour line as a curve that shows where a particular feature of a situation is invariant. When the contour lines make shapes that you recognize, (circles or polygons, for example), you may be able to use the geometry of those shapes to solve optimization problems.

5. (a) You are at an arbitrary point M in a crazy swimming pool with many sides. Figure out the shortest way to get out of the pool. Would you go to a corner or a side?

Pam stood still in the pool and waited for the water to calm down. Then she slapped the surface of the water and watched the ripples . . . .

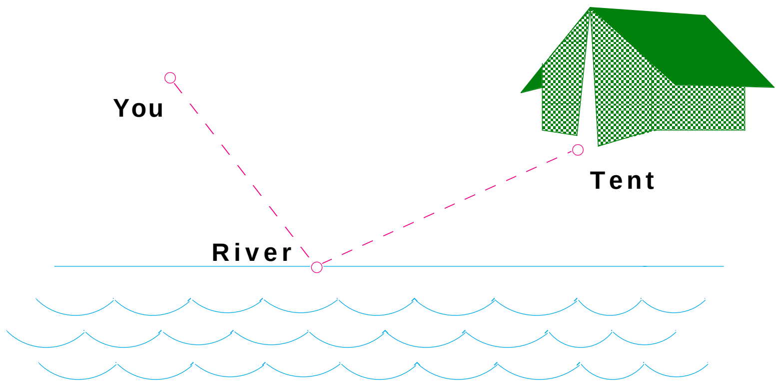


(b) Can you draw a shape for a pool or a position for M so that swimming to a *corner* would be the shortest route out?

Minimizing Distance

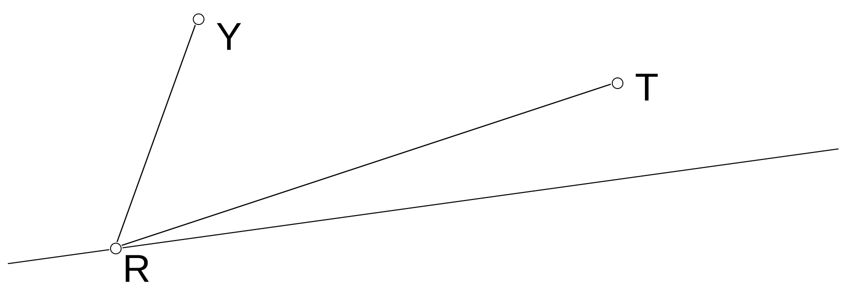
We'll return to the above problem later, but let's take yet another look at the famous burning tent problem.

So, you're on a camping trip. Yet again you see that your tent is on fire. You're still holding the bucket and you're still near the river. And you're still wondering where's the best place to get the water.



What spot minimizes the total travel?

We need to find the point R on the riverbank so that the distance $YR + RT$ (the path to the river to fill your bucket and then to the tent to put out the fire) is minimal.

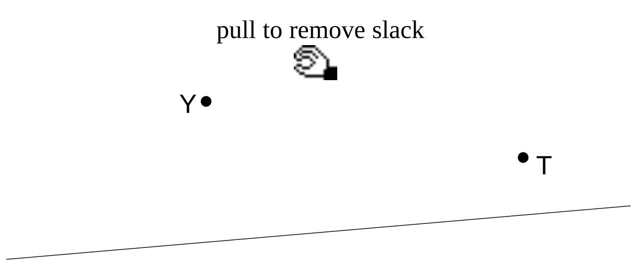


$YR + RT$

The reflection technique could also be used to find a spot that's not *best.*

The pessimist says "equally as bad;" the optimist says "equally as good."

6. Here's a new idea: Pick a point R that you're sure is *not* the best spot and try to draw the shape of all the points on the *plane* (not just along the line of the river) that are "equally as bad" as R. That is, draw every point P with the property that $YR + RT = YP + PT$.



One way to draw such a shape is to do it the "old fashioned" way: stick two pins on your paper where Y and T (you and the tent) are supposed to be and cut a piece of thread as long as the distance $YR + RT$. Tie the two ends of the thread on the pins. Now pull the thread tight with your pencil and trace the thread all the way around, keeping the thread tight all the time.

You can do this as a thought experiment if there's no string around. The string, when pulled taut, just represents a fixed total distance.

Ways to think about it:

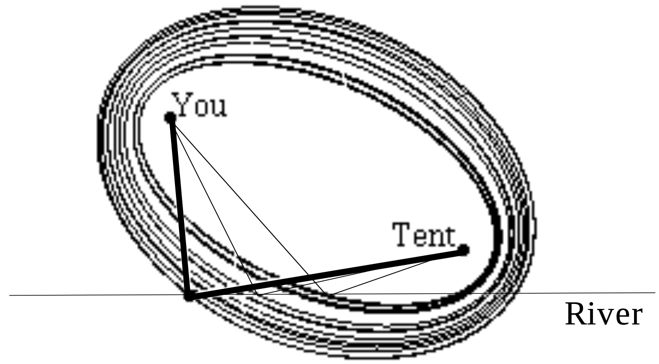
There's something different about problem 6: the domain has changed. Most functions investigated in high school mathematics classes are "number in, number out." That is, they are $\mathbb{R}$ to $\mathbb{R}$ functions. The function in problem 6, however, is defined on the plane; it accepts a point in $\mathbb{R}^2$ and maps it to a value in $\mathbb{R}$. How could you make a sketch of that function? Well, you would want to show several of the kind of curves drawn in problem 6 (the sets of points which produce $YR + RT = k$ for several different values of k)?

Does this sound familiar? Remember your own precalculus course?

You probably recognize these curves as *ellipses*. There are many ways to define ellipses; most often in high school classes they are introduced with conic sections. Here is one formal definition for an ellipse:

> **If A and B are points and k is a number, the set of all points P so that $PA + PB = k$ is called an *ellipse*. A and B are each a *focus* for the ellipse.**

The ellipse that passes through a point R on the shore is a *contour line* for the function that measures the sum of the distances to you and the tent. These are just like the contour lines that weather forecasters and cartographers use—they are lines of constant value. If you look at several contour lines you get a contour plot:

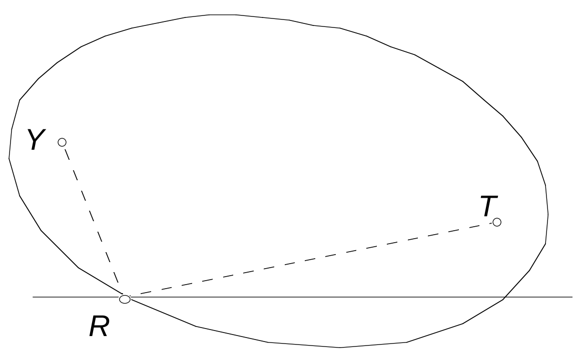


7. Rewrite the definition of ellipses in words you would expect a 10th grade geometry student to use in explaining the concept.

Back to our burning tent problem . . .

If you were precise in constructing the ellipse in problem 6 it most likely crosses the river at two points (one of which is the point R that you picked). If so, then some part of the riverbank lies *inside* the ellipse.

A reasonable distance to run . . .



8. What other possibilities exist for how many intersections the ellipse has with the river? Why does the suggestion that two intersections would be "most likely" make sense?

9. Look at the picture of the ellipse above. If you chose a spot for P where the total distance $YP+PT > YR+RT$, would the ellipse be bigger or smaller than the ellipse shown? How would you compare $YP + PT$ to $YR + RT$ if P were inside the ellipse? Outside?

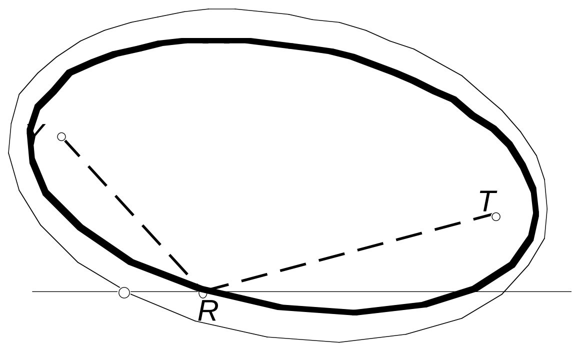
10. Explain this sentence: "As long as there are points along the riverbank that are inside the ellipse, you have not found the optimal spot."

One way to explain it would be to show at least one point that you know *has shorter total distance from you to that point to the tent.*

11. Make a sketch of this situation in dynamic geometry software with R at a non-optimal spot and the ellipse for R constructed. Slide R along the riverbank to a point previously inside the ellipse, and watch the new ellipse form.

⇐Software experiment⇒

Are there any points on the river for which there are *no points inside the ellipse?*



A shorter distance . . .

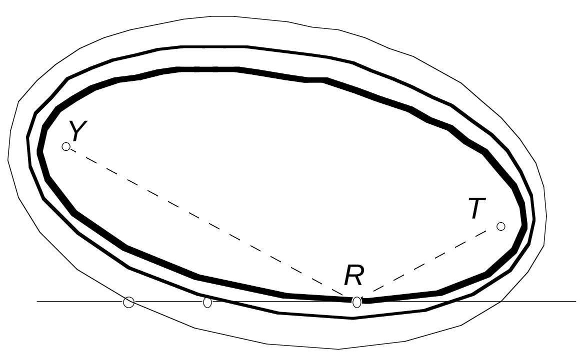
(a) Is the new total distance larger or smaller than the total distance where you initially placed R?

(b) Is the ellipse larger or smaller than the ellipse where you initially placed R?

12. Drag R back and forth along the riverbank. Describe what happens to the ellipses. What do they tell you about the total distance?

Put a trace on the ellipse, and you can see the whole family of ellipses defined by Y and T.

If you continue to slide R along the bank, at some point the ellipse will only touch the riverbank at one point (it will be *tangent* to the riverbank).



The shortest distance . . .

In this picture, there are no points on the riverbank lying inside the ellipse; the distance $YR + RT$ is the smallest we can have. So this is the spot you want to fill your bucket.

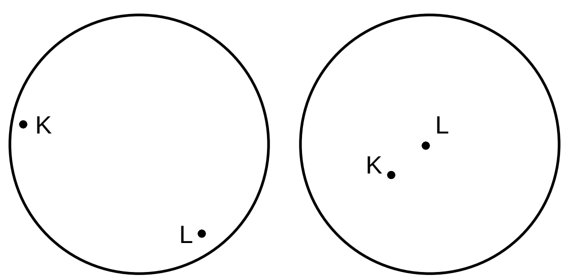
13. Solve problem 5 using contour lines. What shape will your curves of constant value have?

14. In activity 1, you solved this problem using the reflection technique. Try to solve it now using contour lines:

> You're in a rectangular swimming pool at K, out of reach of the sides of the pool. Before swimming to L, you want to swim to a side of the pool to put down your sunglasses. Explain how to find the place to put your sunglasses that minimizes the length of the path you swim.

Questions to Ask: *Could you solve this problem using the reflection principle?*

15. Kris is standing in a circular swimming pool. He's going to swim to some spot in the pool, but first he wants to swim to the edge of the pool to put down his sunglasses. Call his original location K, and the spot where he'll end up L. Explain how to find the best place to put the sunglasses to minimize the total amount of swimming.



Two possible arrangements for this problem

16. Draw a contour plot for the "burning tent" problem. What does the smallest ellipse represent? Is there a largest ellipse for this problem?

There is certainly a largest ellipse you *have drawn, since you can only draw a finite number of them before your next class. But is there a largest one you* can *draw, or is the size unlimited?*

Take it Further

If your tent is on fire, the thing you want to minimize is *time* not distance.

17. Explain why, if you run at a constant speed, the spot on the river that minimizes distance also minimizes the time it takes to get from where you are to the river to the burning tent.

But when Jane Gorman's high school class in Brookline Massachusetts tried to solve this problem, one student complained that the problem ignored the crucial fact that the camper can run faster with an empty bucket than with a full one.

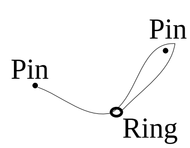
18. Suppose you can run twice as fast with en empty bucket as you can with a full one. Explain why the spot on the river where the ellipse is tangent to the shore is *not* the best place to land. Just using your gut reaction, how would you locate a better spot?

19. In the original burning tent problem, we wanted to find the spot R on the river for which $YR + RT$ is as small as possible. Suppose, as Jane's student suggests, you can run twice as fast with en empty bucket as you can with a full one. Show that the spot on the river that minimizes the total time is the point R where $YR + 2RT$ is a minimum.

Problem 19 suggests that contour lines for the function $R \mapsto YR + 2RT$ might help us solve the "weighted" tent problem in problem 18. Methods for constructing ellipses can be adapted to generate these new curves:

In a new pin-and-string model, tie the string to one pin, lace it through a ring, around the second pin, and tie it to the ring. This doubles the "weight" of the second pin. Now trace the curve by putting a pencil in the ring.

This method extends to allow for curves defined by $PA + kPB =$ constant for any rational k.

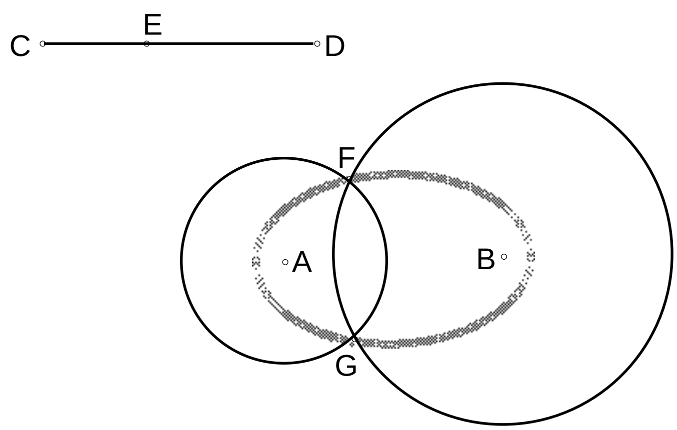


20. Build this pin-and-string model and experiment with it (using different positions of Y and T, for example). Based on your experiments, conjecture how the "best spot" changes when you assume you run twice as fast with an empty bucket.

21. Can you solve this "revised burning tent problem" using the reflection technique discussed earlier? Explain what you need to change in the setup, and any problems you have trying to extend that technique to the situation where you run twice as fast with an empty bucket.

The construction *of this sketch is driven by the fact that the positions of F and G are determined by that of E. And the actual drawing of the ellipse is accomplished by moving E continuously along $\overline{CD}$. In other words, students who construct this sketch are modeling a continuous function and then experimenting with its behavior.*

Using geometry software, you can create ellipses using the intersections of circles. In the picture below, circles of radius CE and ED are constructed with centers A and B, respectively. As E slides back and forth along $\overline{CD}$, points F and G (the intersections of the two circles) trace out the upper and lower halves of an ellipse with foci at A and B and whose major axis has length CD.



22. In the sketch above, $AF + FB$ is a constant. Use dynamic geometry software to make a sketch to construct curves with the property $AF + 2FB$ is constant.

Snell's Law is about refraction of light. Look in a physics book.

23. Look up Snell's Law. Explain how it relates to this "weighted burning tent problem."

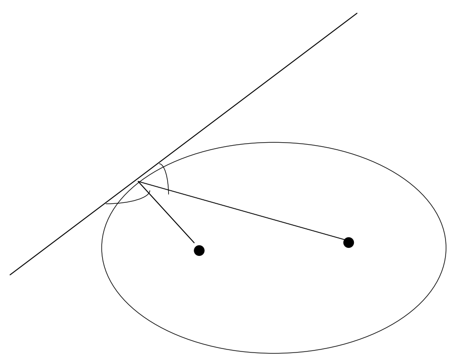
Thinking about students' thinking. Suppose you have a student in your class who comes up with an argument like this for problem 18:

> If I can run faster with an empty bucket, then I want to minimize the mount of time I spend lugging the full bucket. So, just like the "swim and run" problem (problem 1 on page 7), I'd drop a perpendicular from the tent to the river, and that's where I'd land. That minimizes the "heavy lifting," so it would make for the shortest trip."

What would you say to the student?

Checkpoint

24. Show that a tangent to an ellipse makes congruent angles with segments draw to the foci.



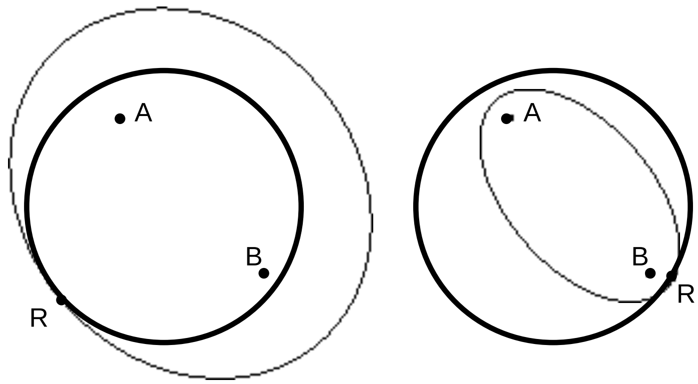
Hint: Look at this as a burning tent problem; compare the contour line solution with the reflection one.

Connections to Calculus

Problem 15 on page 26 was about minimizing total distance from two points to a *circle*. The contour lines of constant distance from two points are ellipses, so in solving the problem you were looking for a spot where an ellipse was tangent to the circle.

Here are two possible ways for an ellipse to be tangent to a circle.

Problem 15: "Kris is standing in a circular swimming pool. He's going to swim to some spot in the pool, but first he wants to swim to the edge of the pool to put down his sunglasses. Call his original location K, and the spot where he'll end up L. Explain how to find the best place to put the sunglasses to minimize the total amount of swimming."



25. Identify which picture shows a true solution to problem 15 and explain why.

26. What does the other picture show? (Explain the point of tangency in relation to the problem of finding the shortest distance from A to the circle to B.)

27. Draw a contour plot for this problem. What does the smallest ellipse represent? Is there a largest ellipse? If so, what does it represent? If not, why not?

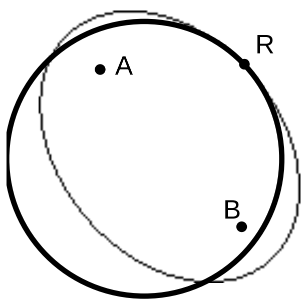
28. Can A and B be positioned so that there are no points that produce the minimum? So that no points produce the maximum?

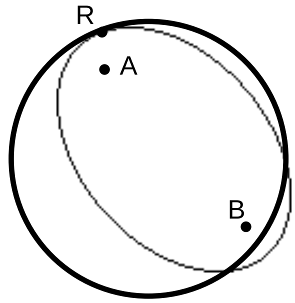
29. Can A and B be positioned so that there is more than one point that produces the minimum?

Set up a sketch like this in dynamic geometry software with a moveable point P and the ellipse for P drawn in. As you move P around the circle, notice how the ellipses grow and shrink. There are some places where they "change direction," where they were shrinking and begin to grow or they were growing and begin to shrink.

30. Look again at the two pictures of tangencies above. Explain how the sizes of the ellipses change as you move P towards and then past the point R in each of these two pictures. Why does this happen?

31. As you move you move P around the circle (depending on how your two points are arranged inside the circle) you may notice other "curious points," like the two pictured below.



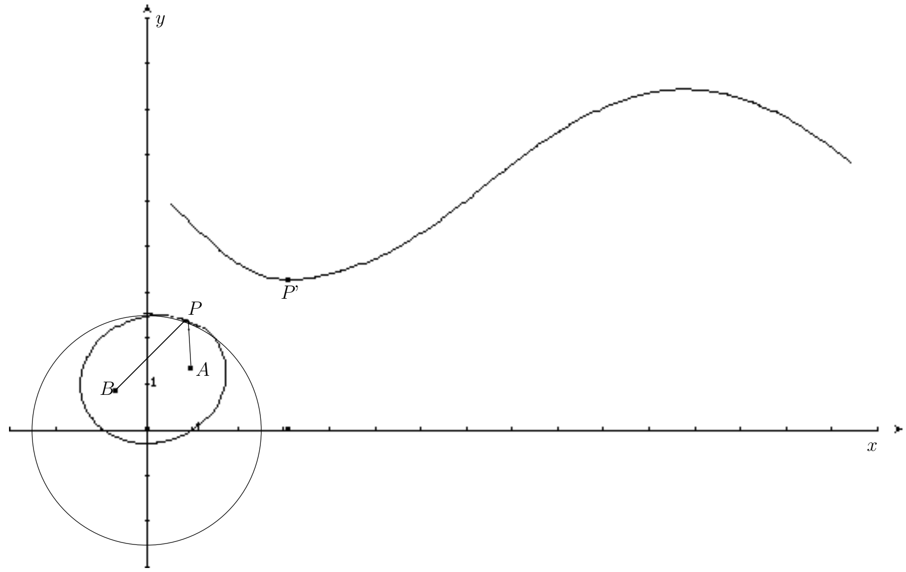


(a) What would happen if you moved P towards and then past the point R in each of these two pictures?
(b) How are these two points different from the maximum and minimum points discussed earlier?

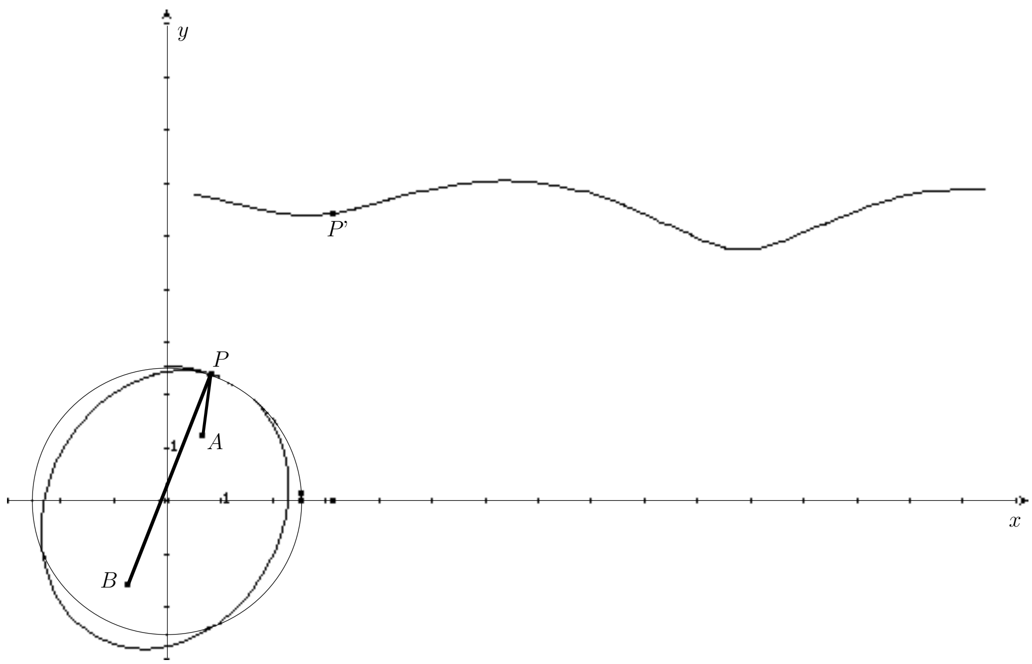
Questions like 28 and 29 ask you to think about the dependence of the function $P \to PA + PB$ on the points A and B. (Prior to that, we were only considering its dependence on P, the point moving along the edge of the pool.) Looking at this function with contour lines (the ellipses) provides us with a useful way to visualize its behavior, but even greater insights can be gained by looking at the interplay between this and a more traditional Cartesian one.

The two pictures below show both a Cartesian graph of the distance of P around the circle (on the x-axis) vs. the total distance $PA + PB$.

You could set up similar sketches in geometry software for more familiar cyclic functions like sine, cosine, and tangent. In all of them the periodicity is clear: as P passes through the intersection of the circle and the x-axis (in the counter-clockwise direction), P' leaves the right end of the graph to return to the left end.



One absolute minimum



Local minimum

32. Create a graph like this using geometry software. Properties of your graph should include:

(a) As you move P around the circle, the ellipse of all points "equally bad as P" also changes dynamically.

(b) The x-axis measures the arc-length from the positive intersection of the circle with the x-axis to the moveable point P.
(c) The y-axis measures the sum of two distances: $PA + PB$.
(d) As you move P around the circle, P' traces out the Cartesian graph, showing you when P is at a local or absolute maximum or minimum.
(e) You can change the positions of A and B, creating different Cartesian graphs.

33. Use your sketch to experiment with the positions of A and B. What positions of A and B give you two absolute minima? What positions give you no relative extrema? State any conjectures you have.

34. What kind of relationship would the ellipse have to the circle at an inflection point?

Write and Reflect:

35. Outline some approaches to contour lines that would be appropriate for
- middle school students
- grades 9–11 geometry students
- grades 10–12 precalculus students
- grades 11–13 calculus students

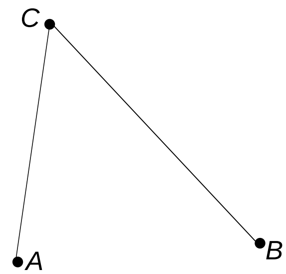
Maximizing Angle

All of the problems so far have been about optimizing distance; the contour lines have mostly been curves of constant distance (circles) or constant total distance from two points (ellipses). Similar techniques can be used for solving problems about optimizing angles.

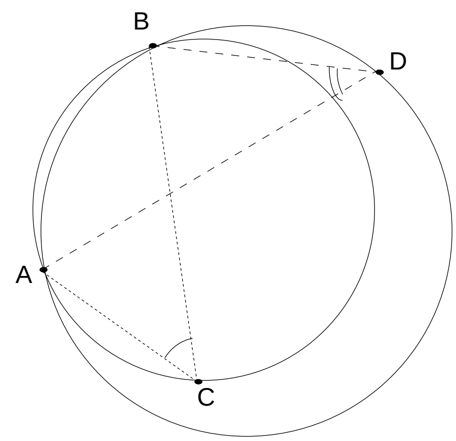
36. Mark two points, A and B, on a piece of paper.
(a) Find a point C so that $\angle ACB$ measures 90°.
(b) Find *all* the points P in the plane so that $\angle APB$ measures 90°.

Can you form a "constant angle tool" or use dynamic geometry software to help you with this problem?

37. Keeping A and B fixed, find all the points P in the plane so that $\angle APB$ is constant and congruent to angle $\angle ACB$ in this picture. What shape curve do you have?



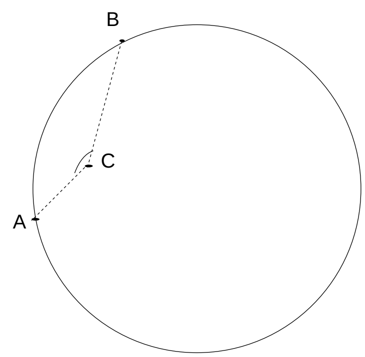
38. Which angle is larger, $\angle ACB$ or $\angle ADB$? Why do you think so?



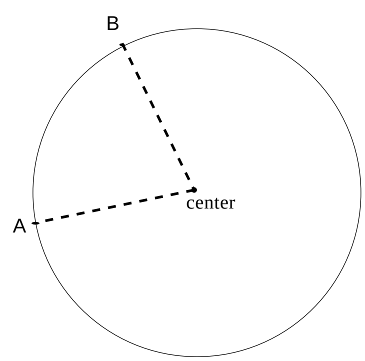
Thinking About Student Thinking:

Two students offered two explanations, both claiming that $\angle ACB$ is larger:

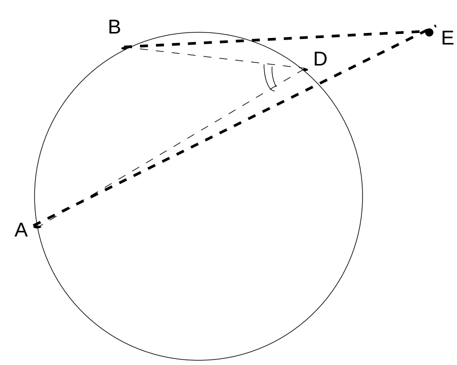
Student 1: I think $\angle ACB$ is bigger because, well, look at the bigger circle. $\angle ADB$ is *on* the bigger circle, and $\angle ACB$ is *inside* the bigger circle. If you start with C really close to A and B like this, it's big.



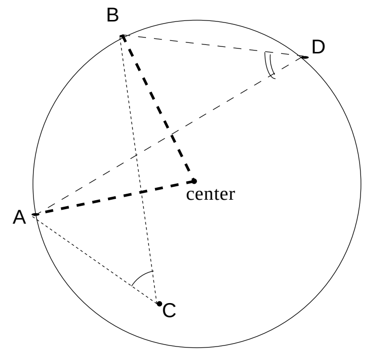
At the central angle it's the same as the intercepted arc. It gets smaller as I move it towards the circle.



The inscribe angle (at D) is half the intercepted arc. Then outside the circle it's even smaller.



Since $\angle ACB$ is between the circle and $\angle ADB$, it's smaller than the intercepted arc, but bigger than half the intercepted arc.

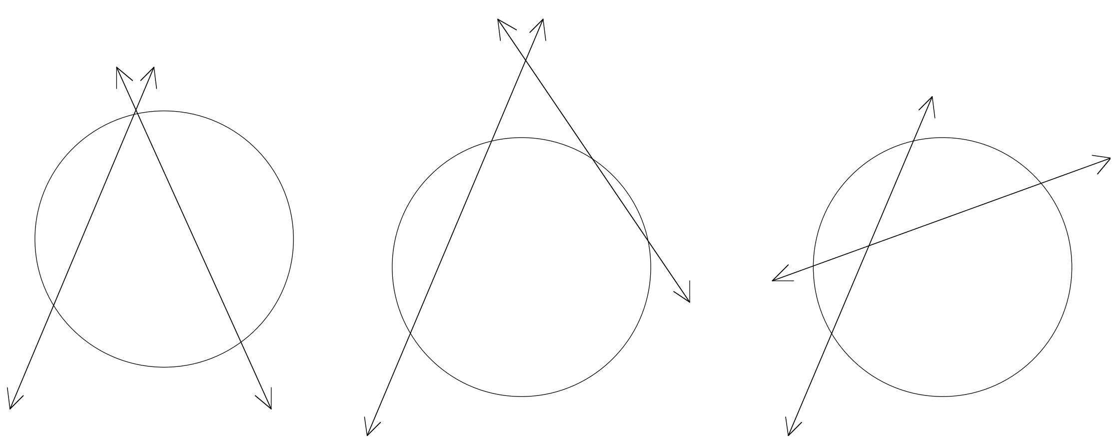


Student 2: I thought of it this way: $\angle ADB$ intercepts an arc on the large circle, and $\angle ACB$ intercepts an arc on the small circle. The arc that $\angle ADB$ intercepts is a smaller percentage of the whole large circle, compared with the arc $\angle ACB$ intercepts on the small circle.

39. What is each student trying to say? Explain both student 1 and student 2's ideas more clearly. What facts from high school geometry are they using?

For Discussion:

There is a whole class of theorems in high school geometry that relate arcs of circles to angles. There are different formulas for each of these:

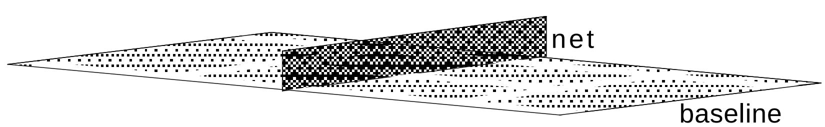


Find out about these theorems, and discuss how they can explored with dynamic geometry software.

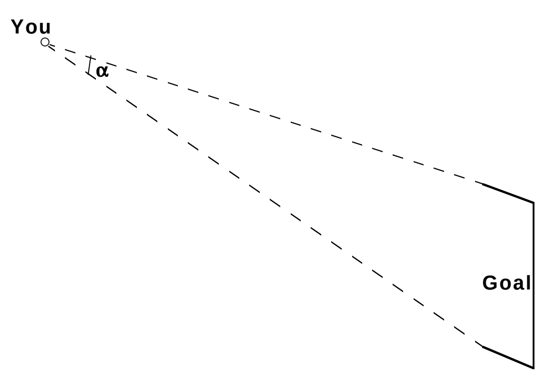
Write and Reflect:

40. What are the curves of constant angle between two fixed points?

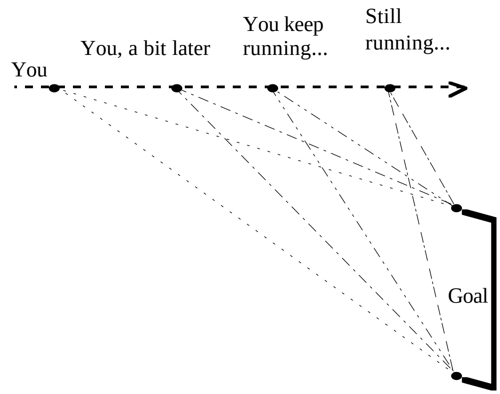
41. In tennis, when you hit from close to the net, you have a wider angle of possible shots than when you hit from the baseline. Using circles, draw a picture that explains why.



42. In soccer, you score by kicking a ball between two goal posts. Suppose you are running straight down the field toward the goal, but off to the side because of the other team's defense. As you run, you have various openings on the goal posts, and thus, opportunities to take a shot. From where should you shoot if you want the widest angle and the best opportunity for scoring a goal?

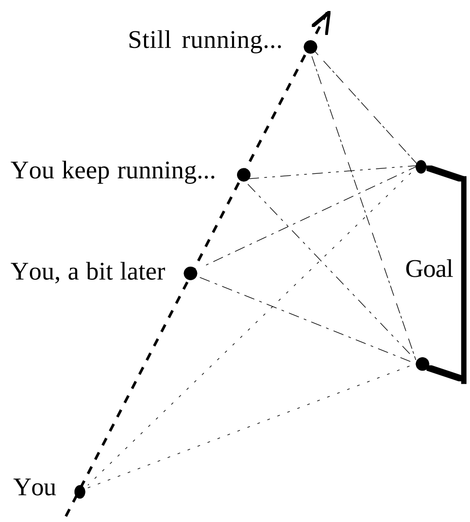


This is your first opening...



...more openings...

43. What if you're playing soccer and you're just running along some straight line (not necessarily perpendicular to the line between the goal posts)? Describe how to locate the spot that maximizes the angle in the picture.



44. You are in an art gallery looking at a picture on the wall. Your eye level is 5 ft. above the ground, the bottom of the picture is 5.5 ft., and the picture is 4 ft. tall and 6 ft. wide. How far from the wall should you stand for the maximum viewing angle? What considerations might influence whether or not this is the best place to stand?

For Discussion:

You began this section by exploring problems posed in a mathematical context (like finding all the points in the plane that keep $\angle ACB$ constant) and went on to solve problems with other contexts (like tennis, soccer, and viewing art). How has the context affected how you worked on these problems?

Part 2. Algebraic Techniques

By high school algebra, we mean the art of transforming algebraic expressions from one form to another. Functions also play an important rôle in high school algebra; we'll study the use of functions in optimization in the next part.

In this part, you'll develop some techniques for solving optimization problems that are based in high school algebra.

1. Squares are never negative

1. Of all the pairs of real numbers that sum to 20, which has the greatest product?

2. Of all the pairs of real numbers that sum to 251, which has the greatest product?

One way to approach these problems is to make a table of number pairs with the requisite sum, and check the products. For problem 1:

x	y	xy
0	20	0
1	19	19
2	18	36
$\vdots$	$\vdots$	$\vdots$

This may be, in fact, how you solved problem 3 in the introductory problems, and it is a valid method when you are limited to integers. There are two problems with using that method here:

- Unless you use a spreadsheet or other technology, the table for problem 2 will be fairly long and tedious to calculate.
- We are no longer dealing with integers. How can we be sure that there aren't two real numbers that sum to 20 with a product greater than 100?

Here we solved for y in terms of x. The other way (solving for x in terms of y) would work just as well.

Another idea, which may lead to a more general method, is to let the numbers be x and y. Then you know that $x+y=20$, and you want to make xy as large as possible. Well, since $x+y=20$, $y=20-x$. So, our product xy is just $x(20-x)=20x-x^2$.

3. Use high school algebra to rewrite this product of $20x-x^2$ as $100-(\text{something})^2$.

Hint: You may want to consider the title of this activity.

4. (*Continuation.*) Use this form of the product to conclude:
(a) That the maximum value for the product of any two real numbers that sum to 20 is 100.
(b) That this maximum value occurs when $x=y=10$.

How could you have come up with the formula $20x - x^2 = 100 - (\text{something})^2$? One way is to know that you want the maximum to be 100, so you start with 100 − something, but you want that something to be positive, so you want it to be a square. Another way is called "completing the square," which you'll learn more about soon. But first, an important theorem:

Theorem 1 *The square of any real number is always greater than or equal to 0.*

5. Use what you know about real numbers to justify this theorem. Is the square of a real number ever equal to 0? When? Can you think of a number system in which the square of a number is negative?

So you have a new principle for solving maximization problems:

> *If you can express a quantity as a constant minus a square, the quantity is never bigger than the constant and is equal to it precisely when the square is 0.*

6. Use this principle to solve problem 2 (if that's not how you solved it the first time).

7. Restate this principle in your own words, and illustrate what you say with an example.

8. State an analogous principle for "a constant plus a square."

Completing the Square

Of course, not every expression can be written as a constant plus or minus a square. But every *quadratic* expression can. Here's how:

Do you remember what a quadratic *expression is?*

The form of $(x + b)^2 = x^2 + 2bx + b^2$ shows that if you square $x + b$, you can get from the coefficient of x (that is, $2b$) to the constant term (b^2), by halving and squaring.

The technique is called "completing the square." Why do you think that is?

So, how could you make $x^2 - 6x$ a perfect square? Well, half of 6 is 3 and $3^2 = 9$, so $x^2 - 6x + 9$ is a perfect square. You only have $x^2 - 6x$, but you can add the 9 and then subtract it (so we

$x^2 - 6x + 9 = (x - 3)^2$

don't add anything to the total), calculating like this:

$$\begin{aligned} x^2 - 6x &= x^2 - 6x + 9 - 9 \\ &= (x-3)^2 - 9 \end{aligned}$$

So, this thing is never smaller than -9, and it equals -9 precisely when $x - 3 = 0$; that is, when $x = 3$.

so, here we have $x^2 - 6x = (x-3)^2 - 9$, a square minus 9.

Here are some examples for you to try the technique:

9. Write each of the following as a constant plus or minus a square. In each case, is there a maximum value or a minimum value for the expression? What is that maximum or minimum value?
(a) $x^2 + 8x$
(b) $12x - x^2$
(c) $7x + x^2$

10. 180 feet of fence is to be used to make a pen in the shape of a rectangle. What size rectangle maximizes the area?

11. You still have those 180 feet of fence to make a rectangular pen, but now one side of the pen will be against a 200 foot wall, so it requires no fence. What size rectangle maximizes the area now?

This is a general version of problems 1 and 2.

12. Show that if two positive numbers add up to a constant, their product is as large as possible when they are equal.

13. Show that if two positive numbers have a constant product, their sum is as small as possible when they are equal.

Hint: Suppose the numbers are r and s and $rs = k$, a constant. To minimize $r + s$, it's enough to minimize

$$(r+s)^2 = r^2+2rs+s^2 = r^2-2rs+s^2+4rs = (r-s)^2+4k$$

But $(r-s)^2+4k$ is smallest when $(r-s)^2$ is smallest.

And squares are never negative.

What is $a \times \frac{25}{a}$?

14. If a is restricted to positive numbers, what is the smallest value of $a + \frac{25}{a}$.

What is the meaning of the distance from a point to a line?

15. A point moves along the graph of $2x+3y = 12$. How close does it get to the origin?

For Discussion:

Overheard in the teachers' room:

Pat: All this stuff about completing the square is completely obsolete now that we have graphing calculators.

Chris: What do you mean?

Pat: Well, look for example, at problem 11. If I let y be the area of the pen and x be the length of one side of the pen that is perpendicular to the wall, then

$$y = x(180 - 2x)$$

So, I graph the function $f(x) = 180x - 2x^2$, find the high point by zooming, and I'm all done.

Chris: But that only gives you an approximation of the best value for x. How would you find the *exact* answer?

Pat: I can zoom as long as I please. Isn't 10 place accuracy enough for most practical applications? And, what would you do if the function came out to be a cubic? For kids who don't know calculus, approximation is the only way.

Suppose you were in the teachers' room. What would you say?

Take it further

An important result in elementary algebra is the *quadratic formula*. It says that the solutions to the equation $ax^2 + bx + c = 0$ are

This is not an optimization problem, but it is an interesting application of completing the square.

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and}$$
$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

16. Prove the quadratic formula.

17. Fran maintains that one can find the minimum or maximum of a quadratic expression by looking half way between the roots. Critique Fran's method.

What's half way between $\frac{-b+\sqrt{b^2-4ac}}{2a}$ *and* $\frac{-b-\sqrt{b^2-4ac}}{2a}$*?*

Wally is looking at Fran's argument in problem 17

Wally: But some quadratic expressions don't have real roots. How can you go half way between two numbers that don't exist?

Fran: Well, they exist,they're just not real numbers. But, to be honest, I'm thinking of the *function* $f(x) = ax^2+bx+c$. It's graph is a parabola, and it's roots are where the parabola cuts the x-axis. Half way between those is the axis of symmetry.

Wally: Well, for functions like $f(x) = x^2 + 4x + 6$, there are no x intercepts. What do you do then?

Fran: Hmmm . . . . I'd flip the parabola over the horizontal line through it's vertex and go half way between the intercepts of the flipped parabola.

Wally: And how do you know that this will give you the same thing as going half way between the roots of the original expression?

18. Suppose $f(x) = ax^2+bx+c$ is a quadratic function whose graph doesn't cross the x axis.

(a) What does this say about the coefficients a, b, and c?

(b) Find the equation of the "flipped" parabola that Fran is talking about. What are it's roots? How are they related to the roots of f?

2. The Arithmetic Geometric Mean Inequality

In this activity, you'll learn an algebraic technique that can be used to solve a surprising variety of optimization problems. But let's start with a problem that can be approached in many ways.

PROBLEM

Find the line that passes through the point $(3, 4)$ that cuts off the smallest area in the first quadrant.

"Find the line" means find any information that will determine the line, like its equation or another point on it.

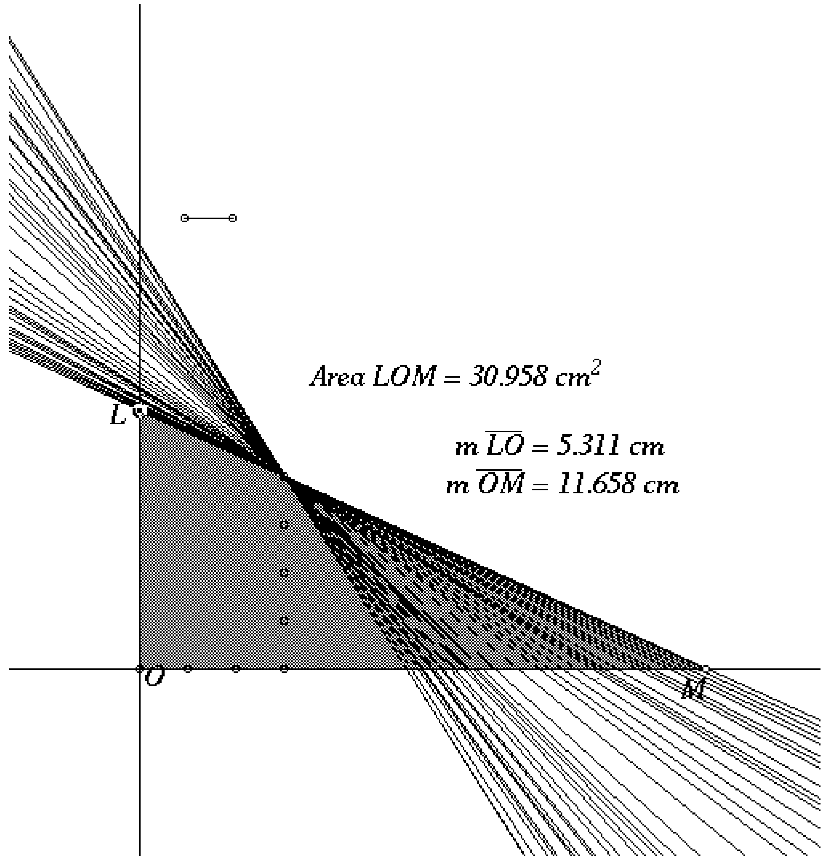
1. Solve the problem any way you can and, whether you come up with an exact or an approximate solution, pay attention to the process and methods you use in solving it. Record some of your thinking about how to solve the problem, and any insights you gained by thinking about the process/methods you used. Make sure to include the different approaches you tried, and which directions or methods failed to help, which seemed most helpful, and why.

Questions to Ask: *What determines the equation of a line? How can you get the intercepts from the equation? Can the intercepts help you solve this problem?*

Ways to think about it:

As usual, the importance of a mathematical problem lies in the *thinking* students develop when working on it. So while working on the following problems we'll also focus on some particular methods used to solve the problem and the mathematical habits of mind behind those particular methods. We'll look at experimental solutions using dynamic geometry software, analytic solutions using calculus for an exact result and graphing for approximate results, a geometric solution, and an algebraic approach that will lead to the main results of this activity.

A dynamic geometry experiment. Most dynamic geometry environments allow you to calculate area of polygons. So, on a coordinate axis, you could set up a "variable" line that passes through the point with coordinates $(3, 4)$, controlled by, say, it's y-intercept.



In this figure, the line passes through $(3, 4)$ and a moveable point L on the y-axis. The system then calculates the area of $\triangle\, LOM$.

2. Set up a similar experiment in your dynamic geometry environment.
   (a) What facts from high school mathematics did you need to know in order to make the set-up?
   (b) Use the set-up to make a conjecture about the optimal line.
   (c) Does the experiment or the design of the set-up convince you of your conjecture?
   (d) Does the experiment or the design of the set-up help you with a proof of your conjecture?

The function can be described by a formula (and we'll do that shortly), but right now, just think of the correspondence $OL \mapsto$ *area* $\triangle\, LOM$

3. The area of $\triangle\, LOM$ is a function of the position of L along the vertical axis; that is, the area of $\triangle\, LOM$ is a function of LO. Without writing anything down, drag L up and down and watch the area calculation change. Describe the behavior of this function. For any area A larger than the smallest area, how many values of OL produce A? What might a Cartesian graph of this function look like?

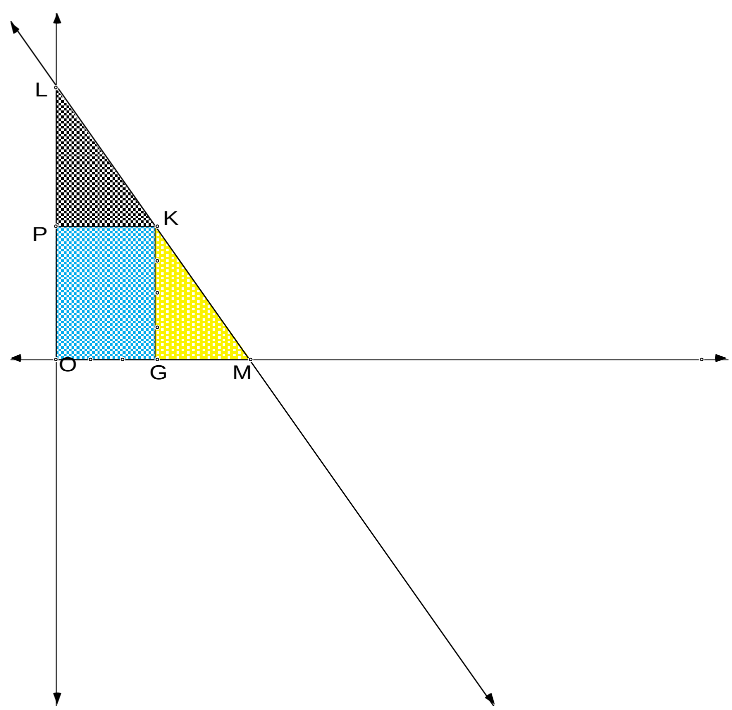
Write and Reflect:

4. What might students learn about the problem be setting up and using a dynamic geometry experiment? How would it affect the way they think about the problem? What mathematics would they use? What mathematics would they learn?

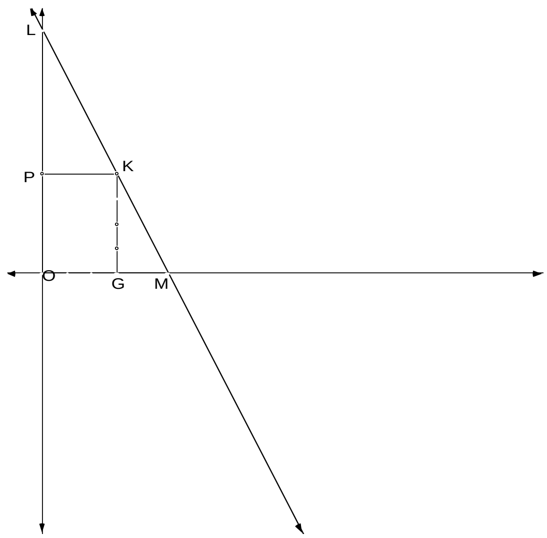
Thinking About Student Thinking:

Joel, an 11th grade student, made the following argument:

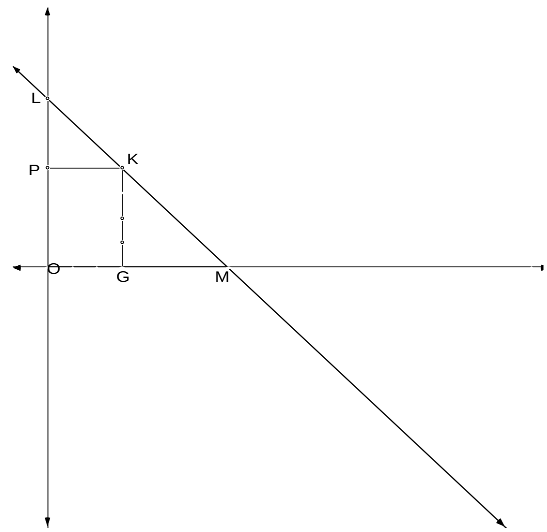
"First of all, there's a rectangle and two triangles.



The rectangle is always 3×4, so it's area is 12. So what I really want to look at are the two triangles, and as I drag L up and down, it's like a seesaw. If the area are not equal, there'll be another place in the seesaw where the areas swap (top to bottom), so the sum remains the same.



area $\triangle LPK = 9$



area $\triangle KGM = 9$

For example, if $LP = 6$, area $\triangle LPK = 9$ and area $\triangle KGM = 4$. But then when $LP = \frac{8}{3}$, area $\triangle LPK = 4$ and area $\triangle KGM = 9$, so they swap. That can't be as small as possible. The only way to not get this swapping is when the two triangles have the same area. That happens when $LP = 4$ and $GM = 3$. "

If two different positions of the seesaw have "top triangle = bottom triangle," do you have to have "bottom triangle = top triangle?"

1. What is Joel talking about?
2. How is he making his calculations?
3. Critique Joel's reasoning.

Functions and graphs. Another way to approach this problem is to use a little analytic geometry. Suppose the equation of the line is $y = ax + b$. Then the sides of the triangle we want to minimize are b and $-\frac{b}{a}$ (problem 7). The area of a triangle is $\frac{1}{2}$ base $\times$ height, so the thing we want to minimize is

$$\frac{1}{2}b \cdot \left(-\frac{b}{a}\right)$$

or

Isn't area supposed to be positive?

$$-\frac{b^2}{2a}$$

Then, because the point with coordinates $(3, 4)$ is on the line, $(3, 4)$ satisfies the equation of the line, so

$$4 = 3a + b$$

and

$$b = 4 - 3a$$

Hence, the area, as a function of a, is given by the function α, where

$$\alpha(a) = -\frac{(4-3a)^2}{2a}$$

Many high school curricula simply assume that students know that a point is on the graph of an equation if and only if its coordinates satisfy the equation. Even though this is true by definition, it's not so clear to many students.

5. Investigate the function α from the point of view of a pre-calculus student.

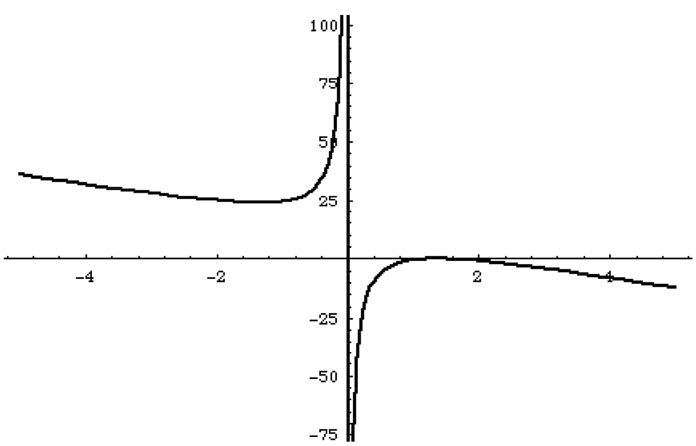
(a) What is its "natural domain" (the domain imposed by the context of the problem)?

(b) Are there asymptotes to the graph?

(c) Approximate the minimum value and the value of a that produces it using a numerical technique or a zooming feature.

Thinking About Student Thinking:

Dee produced the following graph of α in her computer algebra system:



She said

I know a has to be negative, but the graph is sort of the same on either side of the vertical axis, so I'll look at positive values of a. Positive as always produce negative αs, because a is the slope of the line and when that's positive, the line doesn't cut off any area in the first quadrant but it does cut of negative area in some other quadrant. But α is *zero*, when $a = \frac{4}{3}$ because that makes the numerator of the fraction in the formula for α. So, that place where it looks like the graph just touches the horizontal axis is at $a = \frac{4}{3}$, and it really is tangent, so it's a maximum value, and so, by symmetry, the minimum value is at $a = -\frac{4}{3}$."

1. What is Dee talking about?
2. Is her reasoning "by symmetry" justified?
3. What would you say to Dee if you were her teacher, and you discussing her solution with her?

When is α 0? When is it infinite?

6. Investigate the function α using calculus.
(a) What is its "natural domain" (the domain imposed by the context of the problem)? Is there any interpretation for regions of the graph that don't correspond to the situation we're investigating?

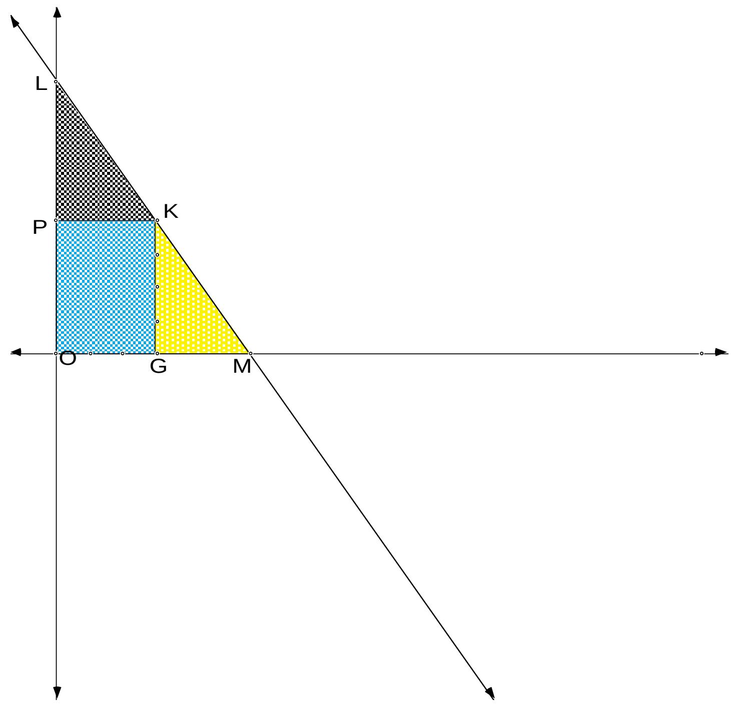
(b) Find all extreme values of α using differentiation.

7. Show that if a line with equation $y = ax+b$ passes through a point in the first quadrant and makes a triangle with the axes, then
(a) $a < 0$ and $b > 0$
(b) the triangle has legs of length b and $-\frac{b}{a}$.

If $a < 0$ and $b > 0$, then $-\frac{b}{a}$ is positive.

An algebraic approach. Let's take a closer look at the setup Joel used in his argument on page 47.

"First of all, there's a rectangle and two triangles.



The rectangle is always 3×4, so it's area is 12. So what I really want to look at are the two triangles ...."

Our problem asks us to minimize the area of $\triangle LOM$. Joel breaks this triangle up into a rectangle of constant area (12) and two triangles, $\triangle LPK$ and $\triangle KGM$, whose areas change as L moves up and down the vertical axis. So, to minimize the area of $\triangle LOM$, it's enough to minimize the sum of the areas of $\triangle LPK$ and $\triangle KGM$. If you try the experiment on page 46 (either as a thought experiment or on your computer), you see that these two right triangles each have one leg whose length is fixed and another whose length changes. Let $LP = r$ and $GM = s$. Then

Make sure you can explain this. How is the area of $\triangle LOM$ connected to the sum of the areas of $\triangle LPK$ and $\triangle KGM$?

$$\text{area}(\triangle LPK) = \frac{1}{2}\,3r \quad \text{and}$$
$$\text{area}(\triangle KGM) = \frac{1}{2}\,4s$$

So, we want to minimize

$$\frac{1}{2}\,3r + \frac{1}{2}\,4s = \frac{1}{2}(3r + 4s)$$

For this, it's enough to minimize $3r + 4s$. Now, if r and s were "independent" and free to take on any values, there'd be no hope; you could make $3r + 4s$ as big as you wanted just by

Joel's idea again: Here we "ignore" the constant $\frac{1}{2}$. Why is it "enough" to minimize $3r + 4s$?

making r and s big enough. But r and s are related to each other, as the next problem shows:

8. Show that

$$\triangle LPK \sim \triangle KGM$$

And using this, show that $rs = 12$

Ways to think about it:

So, we want to minimize $3r+4s$ subject to the *constraint* $rs = 12$. Enter problem 13 on page 42. It says, "If two positive numbers have a constant product, their sum is as small as possible when they are equal."

Certainly, r and s have a constant product, but we don't want to minimize their sum $r + s$. We want to minimize $3r + 4s$. But wait; if $rs = 12$, isn't $(3r)(4s) = 144$? So, if we think of our "positive numbers" as $3r$ and $4s$, our problem is to minimize their sum, knowing that their product is constant. Problem 13 on page 42 says that the sum is minimized in this situation when the factors are constant. That is, $3r + 4s$ is a minimum when $3r = 4s$. But since $(3r)(4s) = 144$, the summands are equal precisely when $3r = 4s = 12$. Hence $r = 4$ and $s = 3$.

What two positive numbers multiply to 144 and are equal?

9. So, what's the equation of the line that passes through $(3, 4)$ and cuts of minimal area in the first quadrant?

10. More generally, what's the equation of the line that passes through (ℓ, m) and cuts of minimal area in the first quadrant?

A general class of problems. We're onto something quite general here. We have two "dual" classes of problems: if r and s assume only positive values,

ℓ and m are assumed positive here.

1. minimize $\ell r + ms$ subject to the constraint $rs = k$, and
2. the dual problem: maximize rs subject to the constraint $\ell r + ms = k$

Using problems 12 and 13 on page 42, you can prove the following theorem:

Theorem 2 *Suppose r and s assume only positive values. Then*

1. *if $rs = k$ (a constant), then $\ell r + ms$ is a minimum when $\ell r = ms$*
2. *if $\ell r + ms = k$ (a constant), then rs a maximum when $\ell r = ms$*

ℓ and m are positive constants.

11. Prove this theorem.

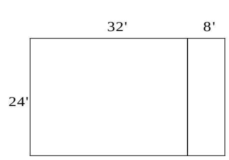
Actually, the theorem is true for more than two variables, and the general result can be used to solve all kinds of optimization problems. The goal of the rest of this activity is to get to this more general result, and we'll do it via a very beautiful theorem called the "arithmetic geometric mean inequality." Let's see what it says, first for two variables and then for any number.

Remember problem 2 on page 15?

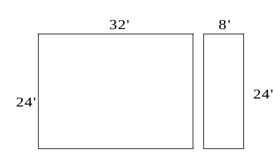
> The expense of building a house depends greatly on its perimeter; that is where the expensive things like windows and doors are found. Suppose you want to build a house with a rectangular base. Given your budget, you decide you can afford a house with total perimeter of 128 feet. What dimensions should you choose for the base of the house if you want to maximize its area?

And Jo's solution to it?

> I'll show that a 32×32 square is best by demonstrating that any other rectangle with a perimeter of 128 has area smaller than the area of the square. I'll do this by showing that I can cut up such a rectangle and make it fit inside the 32×32 square with room to spare.
> Suppose, for example, I have a 40×24 rectangle.
> First, I'd cut it like this:

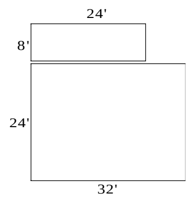


Snip.

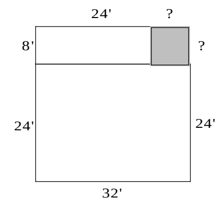


Then rearrange these

> Then, I take the small strip off the side and put it on top of the 24×32 rectangle:



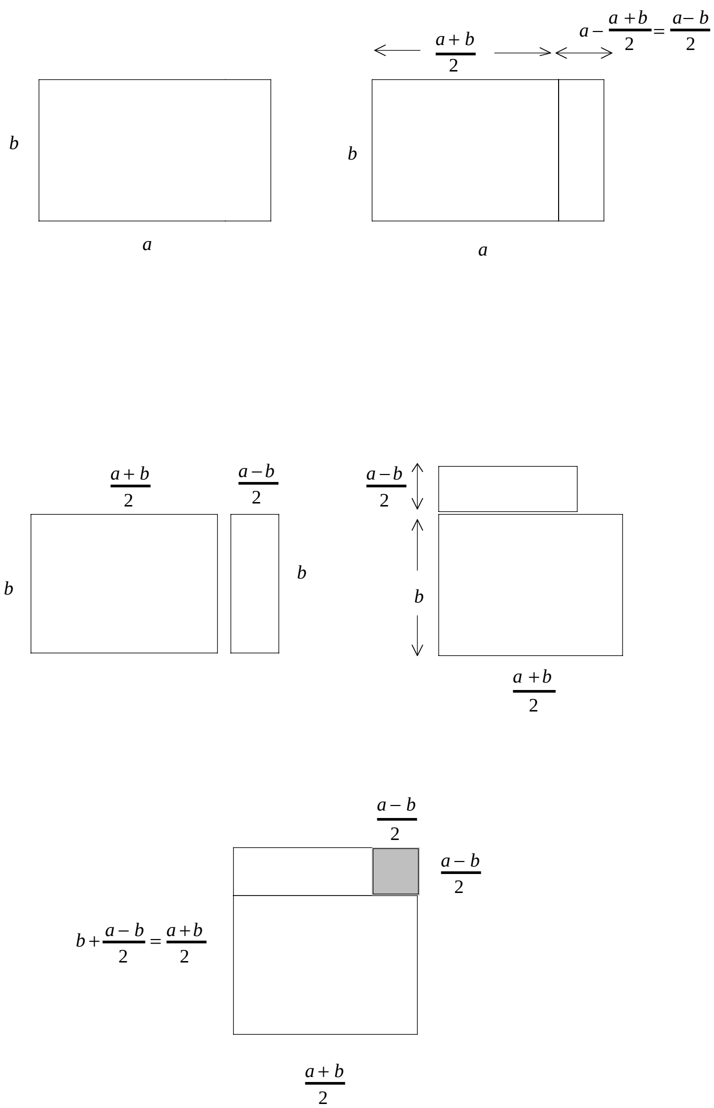
to get this



Which is not quite a square.

> The operation *rearranged* the area, but I didn't add or lose any area. Meanwhile, my two pieces cover *some* of the 32×32 square, but not all of it. The shaded part isn't covered, so the square has more area. Therefore, the area of the 32×32 square is *bigger* than the area of the 40×24 rectangle.

One purpose of algebra is to turn numerical calculations like this into *generic* calculations that work for all numbers. That's what you did in problem 5 on page 17. Your argument probably used pictures like this:



Write and Reflect:

12. Explain each frame in this picture.

13. Explain how the picture can be interpreted to show that

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

In Mathematics Magazine *there is a feature called "Proof without Words." Is the above picture a proof without words?*

14. In problem 13 you used a picture as motivation for an algebraic identity:

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

Prove the identity using only algebra.

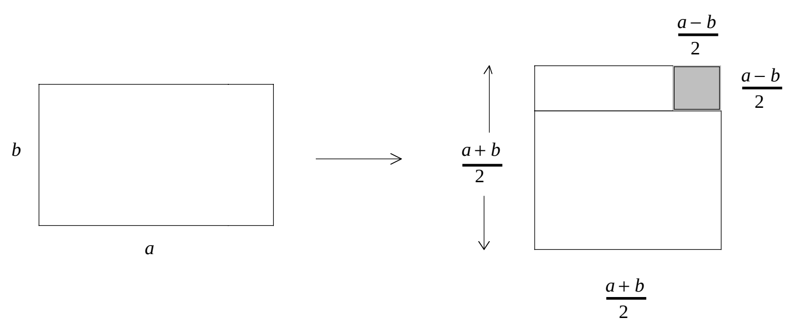
A conversation between two people

Oscar: I only understand something when I can visualize it. Unless I can see a picture of something, it makes no sense to me.

Claudia: What do you mean by "picture?" Is every mental image a picture? What do you see when you think of

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

Oscar: I see this (writing on the board):



Surely, Claudia, when you think of the identity, you see *something*.

Claudia: Yes, I see this (writing on the board):

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

For Discussion:

What exactly is the rôle of visualization in mathematics? Are algebraic expressions less visual than geometric figures? More abstract? What are the limitations of a "visual" explanation? Do you think there are students for whom geometric figures are a hindrance?

Thinking About Student Thinking:

Joon explains the identity

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

this way:

"Look at the right side. If you expand the binomials, you get two fractions, each with denominator 4. So, you can combine the numerators. But if you do that the square terms cancel, the middle term of the second binomial changes sign, and it combines with the first middle term, so you get $4ab$. Over 4. So, it's ab."

Would you accept Joon's explanation? What would you say about it?

The identity $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$ can be exploited in many ways.

Often, when you look at an algebraic identity, you can find "hidden meanings."

15. Is $(\frac{a-b}{2})^2$ ever negative? Is it ever zero? Explain your answers.

16. Based on your answer to problem 15, show that, if a and b are non-negative numbers,

$$ab \leq \left(\frac{a+b}{2}\right)^2$$

where equality holds if and only if $a = b$.

The inequality you devised in problem 16 turns out to be very important in mathematics, and it can be used to solve all kinds of optimization problems. It's so important it's given a name:

Theorem 3 ***The Arithmetic Geometric Mean inequality***. *If a and b are non-negative real numbers,then*

$$\sqrt{ab} \leq \frac{a+b}{2}$$

If a and b are not negative, both sides of this inequality are not negative, so you can take the square root and preserve the inequality:

$$\sqrt{ab} \leq \sqrt{\left(\frac{a+b}{2}\right)^2} = \frac{a+b}{2}$$

This is the standard way to write the inequality.

Let's abbreviate the theorem by calling it the "AGM."

Before we see why it's such a big deal, let's look at the reasons behind its name:

The *arithmetic mean* of two numbers a and b is their average. It is the number m that makes the sequence

$$a, m, b$$

an *arithmetic* sequence ($m - a = b - m$).

What's so geometric about the geometric mean? Or about geometric sequences? See if you can find examples from secondary mathematics that explain why the term "geometric" is used for the geometric mean and geometric sequences .

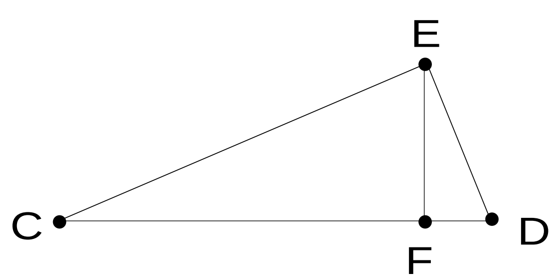
The *geometric mean* of two numbers a and b is the number r that makes the sequence

$$a, r, b$$

a *geometric* sequence (so that $\frac{a}{r} = \frac{r}{b}$).

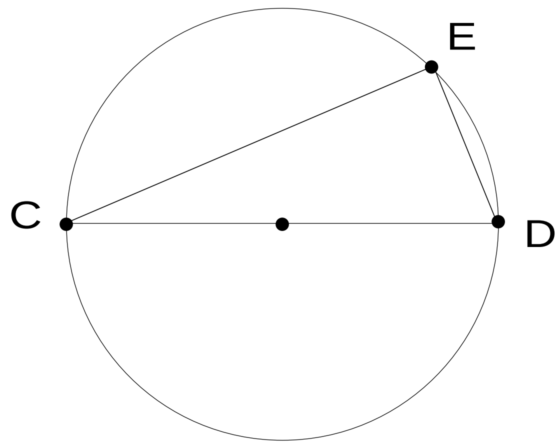
The AGM simply says that the geometric mean is never bigger than the arithmetic mean. Another way to think about the AGM is to use some facts from high school geometry.

17. Prove that the altitude to the hypotenuse of a right triangle is the geometric mean between the segments into which it divides the hypotenuse.



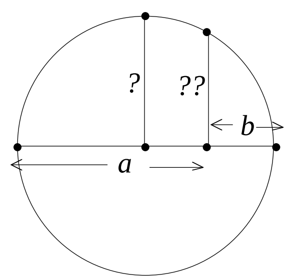
$$\frac{CF}{EF} = \frac{EF}{FD}$$

18. Prove that an angle inscribed in a semi-circle is a right angle.



$\angle E$ is a right angle

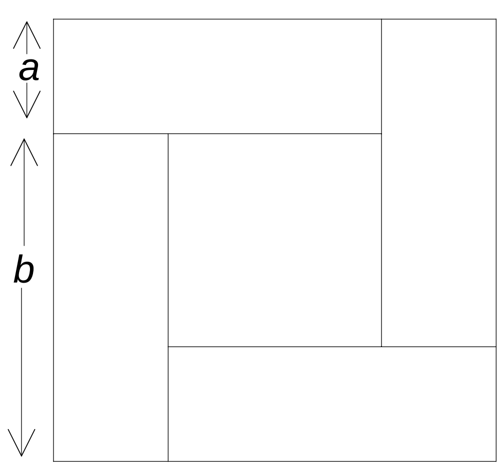
19. Use this figure to provide another proof of the AGM:



How big is the radius of the circle?

20. Using problem 19, when are the two sides of the AGM equal? That is, when is $\sqrt{ab} = \frac{a+b}{2}$? Explain.

21. Use this figure to provide another proof of the AGM:



Hint: Four rectangles of area ab don't quite fill up an $a+b$ square.

22. Using problem 21, when are the two sides of the AGM equal? That is, when is $\sqrt{ab} = \frac{a+b}{2}$? Explain.

Given the results of problems **??**, 20, and 22 we can strengthen the AGM like this:

Theorem 4 ***The AGM: New improved version.*** *If a and b are non-negative real numbers, then*

$$\sqrt{ab} \leq \frac{a+b}{2}$$

and the two sides are equal if and only if $a = b$.

Using the AGM

The AGM is an *inequality* and inequalities can often be used to solve optimization problems, especially if you know the conditions under which both sides are equal.

Thinking About Student Thinking:

Jake was working on problem 1 from the previous activity:

> Of all the pairs of numbers that sum to 20, which has the greatest product?

Jake reasoned like this:

> If the numbers are a and b, then $a + b = 20$ and I want to maximize ab. Hmmm. Sum and product makes me think AGM. I know that $\sqrt{ab}$ is never bigger than $\frac{a+b}{2}$. But
>
> $$\frac{a+b}{2} = \frac{20}{2} = 10$$
>
> so $\sqrt{ab}$ is never bigger than 10. That means that ab is never bigger than 100.
> Is it ever equal to 100? The AGM says that $\sqrt{ab} = 10$ precisely when $a = b$. So the largest value of the product ab is 100, and this happens when a and b are equal, so when $a = b = 10$. Done.

23. Critique Jake's reasoning. Is it a correct application of the AGM?

24. How would Jake solve this problem?

Of all pairs of numbers that multiply to 81, which two have the smallest sum?

25. If you want to build a rectangular pen with area 100 sq. ft., what's the least amount of fencing you'll need to use? What shape will the pen be if you use the minimal amount of fencing?

Problem 25 is "the same" as one you've already solved. Which one? In what way are they the same?

26. Prove the following optimization theorem using the AGM:

Theorem 5 *Suppose a and b are restricted to be non-negative.*

(a) If the sum $a+b$ is constant, the product ab is a maximum when $a = b$.

(b) If the product ab is constant, the sum $a+b$ is a minimum when $a = b$.

These are essentially problems 12 and 13 on page 42. Solve them here using the AGM.

27. Give a proof of theorem 2 on page 53 that uses the AGM.

28. Use theorem 6 to solve the following problem:

Find the maximum value of CN if $10C+12N = 24$

C might stand for "cost" and N for "number sold."

29. On a trip of 100 miles, a trucking company knows that the fuel cost in dollars will be about $\frac{1}{5}$ of the average speed in miles per hour. If the company pays its drivers \$18 per hour, what average speed will minimize the cost of the trip? What will the least expensive trip cost?

The cost of the trip is the cost of gas plus the driver's wage. Is the assumption about the fuel cost reasonable?

30. Show that the sum of a positive number and its reciprocal is never smaller than 2.

31. Suppose

$$f(x, y) = (x + y)\left(\frac{1}{x} + \frac{1}{y}\right)$$

What is the minimum value that f takes if x and y are positive numbers?

32. The area of an ellipse is πab where a and b are half the lengths of the major and minor axes. Suppose the sum of the lengths of the major and minor axes of an ellipse must be 100. Which ellipse has the largest area?

Can you prove that the area of an ellipse is πab? Does this formula make sense?

33. If $f(r, s) = 2rs$, what is the minimum value taken on by f if $6r + 4s = 12$

Thinking About Student Thinking:

Nicky has another proof of the AGM:

Start with an obvious fact: $(a-b)^2 \geq 0$
work this out: $a^2 - 2ab + b^2 \geq 0$
Add $4ab$ to both sides: $a^2 + 2ab + b^2 \geq 4ab$
Factor the left side: $(a+b)^2 \geq 4ab$
divide by 4: $\frac{(a+b)^2}{4} \geq ab$
Square root both sides: $\frac{a+b}{2} \geq \sqrt{ab}$

Write and Reflect:

34. Explain and critique Nicky's work.

If you know about vectors, does $ac + bd$ look familiar?

35. Suppose (a, b) and (c, d) are two points on the unit circle. What can you say about the possible values of $ac + bd$?

Hint: Since all numbers are positive, numbers,

$$a^2 + c^2 \geq 2ac \quad \text{and} \quad b^2 + d^2 \geq 2bd$$

Add these inequalities.

More than two variables As promised on page 53, we can extend the AGM and it's uses in optimization to more than two variables. Let's start with four variables (we'll come back to three in a minute).

And, in general, the arithmetic mean is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

and the geometric mean is

$$\sqrt[n]{a_1 a_2 \cdots a_n}.$$

Suppose you have four numbers a, b, c, d. The arithmetic mean is defined as $\frac{a+b+c+d}{4}$. The geometric mean is defined as $\sqrt[4]{abcd}$.

36. Suppose you start with a sequence of numbers a, b, c, d, but you replace a and b by two numbers equal to the arithmetic mean $\frac{a+b}{2}$. What happens to the sum of the four numbers? (Does it increase, decrease, or stay the same compared with the original sum?) What happens to the product of the four numbers? Explain your answers.

Hint: Use the result of problem 36 on a, b and on c, d. Then use the AGM for two variables.

37. Prove the AGM for four variables? That is, can you show

$$\sqrt[4]{abcd} \leq \frac{a+b+c+d}{4}$$

When does equality occur?

38. Show that

$$\sqrt[4]{abc\sqrt[3]{abc}} = \sqrt[3]{abc}.$$

39. The AGM inequality for three variables says that if x, y, and z are any three positive numbers, then

$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3}$$

Prove the AGM for three variables. When does equality occur?

Hint: Use the result of problem 37. Let $d = \sqrt[3]{abc}$, the geometric mean of the three numbers.

40. If you were going to prove the AGM for n variables using the technique outlined in this problem set, what case would you tackle next? What would be your overall strategy?

Write and Reflect:

41. State and prove an AGM for any number of variables and check it out numerically. There are many classical ways to prove the theorem. One is suggested in the previous problem set. Another is outlined in Camille's method below. Still another is due to Augustin Cauchy (1789-1857). If you get stuck, go to a library and look up a proof, rewriting it in your own words and illustrating the proof with examples.

Thinking About Student Thinking:

Camile has an algorithm for proving the AGM in any number of variables. She illustrates with an example:

"Suppose my numbers are $\{1, 3, 4, 8, \text{ and } 9\}$. I replace the smallest by the average of all the numbers and the largest by the sum of the smallest and largest minus the average of all the numbers, keeping the sum constant and increasing the product. Then I keep doing that, like this:”

$$\begin{aligned} 1 \cdot 3 \cdot 4 \cdot 6 \cdot 9 &< 5 \cdot 3 \cdot 4 \cdot 8 \cdot (1 + 9 - 5) = 5 \cdot 3 \cdot 4 \cdot 8 \cdot 5 \\ &< 5 \cdot 5 \cdot 4 \cdot 6 \cdot 5 \\ &< 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \\ &= 5^5 \end{aligned}$$

42. Try Camile’s method for $\{2, 3, 7, 8, 10, \text{ and } 12\}$

43. Does Camille’s method always work? If so, give a proof. If not, explain why.

44. Prove the following optimization theorem:

Theorem 6 *Suppose a, b, and c are restricted to be non-negative.*

(a) If the sum $a + b + c$ is constant, the product abc is a maximum when $a = b = c$.

(b) If the product abc is constant, the sum $a + b + c$ is a minimum when $a = b = c$.

45. Generalize the result of problem 44 to n variables.

Here is a common problem in secondary mathematics curricula, also suggested in the NCTM *Standards*:

PROBLEM

The Jar Problem. You are asked to manufacture a cylindrical glass jar that has an open top. What are the proportions of the least costly jar which holds a given amount?

The assumption is that the cost of manufacturing a jar is only dependent on the surface area of the jar (how much material is used). Student investigations usually include data-gathering, writing equations, and graphing. Could high school students solve this problem exactly?

46. (a) What is the volume of a cylinder in terms of the radius of the base and the height?

(b) What is the surface area of a cylinder in terms of the radius of the base and the height? (Remember, it’s a jar so it has no “top.”)

(c) According to the problem, which of these will we be holding constant? Which are we trying to minimize?

Write and Reflect:

47. Investigate this problem any way you choose. Make several cylinders of the same volume and different surface area, graph an equation, or do something else to come up with a conjecture. Describe how you investigated the problem, what you concluded, and how certain you are of your conclusions.

---

48. (a) Show that you can write the surface area for the jar as $S = \pi rh + \pi rh + \pi r^2$.
(b) Show that, in the context of this problem, the product $(\pi rh)(\pi rh)(\pi r^2)$ is a constant.
(c) From problem 44, what can you say about minimizing the sum $S = \pi rh + \pi rh + \pi r^2$?

Another optimization problem that high school students often see:

PROBLEM

The Triangle of Fixed Perimeter Problem. Of all triangles with a fixed perimeter, which one has the most area?

You've already solved the analogous problem for rectangles, and you can probably guess at the answer here. But can you solve it with the AGM?

A useful fact for solving this problem relates the area of a triangle to its sidelengths : The standard formula for the area of a triangle is $A = \frac{1}{2}bh$ or, "half the base times the height." Another area formula, known as "Heron's formula," involves only the three sidelengths a, b, c:

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \frac{1}{2}(a+b+c).$$

s is called the "semiperimeter" of the triangle.

Write and Reflect:

49. Look up a proof of Heron's formula, or come up with your own. Are you convinced it works? How can triangles of the same perimeter have different areas at all, if area depends only on sidelengths?

50. Investigate the "triangle of fixed perimeter" problem in any way you choose: with dynamic geometry software, using data collection, etc. Write up your conjectures.

---

51. (a) In Heron's formula applied to this problem, what values are fixed and what values vary?

(b) Use algebraic manipulations to rewrite Heron's formula in the form $\frac{A^2}{s} = (s-a)(s-b)(s-c)$.

(c) Show that the sum $(s-a)+(s-b)+(s-c)$ is constant in this problem.

(d) From problem 44, what can you say about maximizing the product $\frac{A^2}{s} = (s-a)(s-b)(s-c)$? Does this maximize the area of the triangle? What kind of triangle is it?

Thinking About Student Thinking:

Susan claims she can solve the triangle with fixed perimeter problem without any fancy machinery. She says,

"Suppose the fixed perimeter is some number, like 10. Suppose you show me the best possible triangle, the one with the most area. If two of its sides are not equal, I can make the area bigger by keeping the third side constant and thinking of the other two sides as a piece of string. I can move my pencil, like I was making an ellipse, until the two sides *are* equal. That will keep the third side the same (which I'm thinking of as the base), keep the perimeter the same, and increase the height of the triangle, making the area bigger. I guess I'm saying that if two sides aren't equal, I can tweak the triangle and not change the perimeter but make the area bigger. So the best triangle has to have all sides equal."

52. What is Susan trying to say? Is there any merit to her argument? Does it contain any holes?

Checkpoint: Some Miscellaneous problems

Here are some problems that you can solve any way you want.

53. What is the maximum value of xyz as (x, y, z) roams around the plane with equation $3x + 4y + 7z = 12$?

54. What is the maximum value of xyz as (x, y, z) roams around the plane with equation $ax + by + cz = 12$?

55. A point moves along the graph of $xy = 8$ in the first quadrant. What is the minimum value of $3x + 4y$?

56. Show that the sum of n positive numbers times the sum of their reciprocals is always greater than or equal to n^2. When is this product exactly equal to n^2?

57. Show that if x, y, and z are non-negative,

$$xyz \leq \frac{x^3 + y^3 + z^3}{3}$$

When are both sides equal? Generalize to n variables.

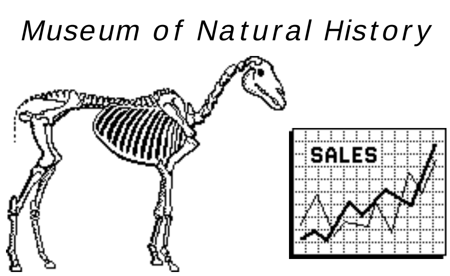
Part 3. Graphical Techniques

In the previous sections you developed geometric and algebraic techniques for solving optimization problems. However, some optimization problems can't be solved using only geometry and algebra; for these we will develop some graphical techniques and habits of mind (like reasoning by continuity) that enable you to find *approximate* solutions.

You could use calculus to find exact solutions to these problems, but most secondary students can't. These problems provide a bridge from thinking about optimization problems they can solve exactly to ones that require a different set of tools they don't have yet.

58. The admission price for the Museum of Natural History is \$8.25. An average number of 450 people visit the museum daily. The Museum recently added a new exhibit, and is thinking of raising the admission price. The management noticed that whenever they raised the price in the past, fewer people visited the museum. More precisely, for each additional 25 cents, 15 fewer people came to visit each day. Which admission price would maximize the museum's income in a day?

Questions to Ask: *Is it true that the highest admission price will bring in the most money to the museum? Or will cheaper tickets cause more people to visit, and increase the museum's income?*



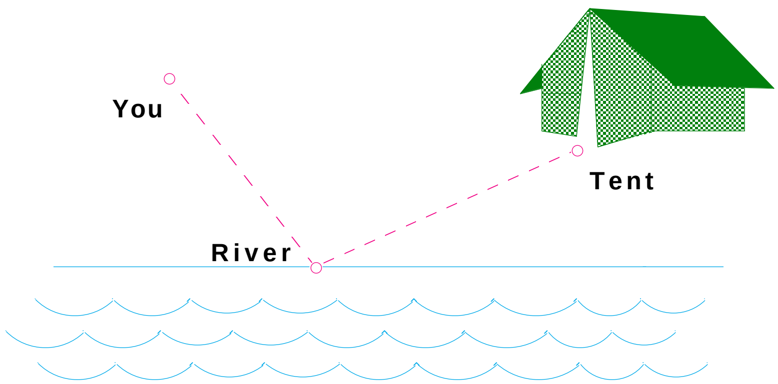
59. A piece of wire 16" long is cut into 2 pieces. One piece is bent to form a square and the other is bent to form a circle. Where should the cut be made for the sum of the areas to the square and circle to be minimum? Maximum?

Questions to Ask: *Have you considered any extreme cases?*

You probably remember the "burning tent" problem:

PROBLEM

You're on a camping trip. While walking back from a hike, you see that your tent is on fire. Luckily you're holding a bucket and you're near a river. Where should you get the water along the river to minimize your total travel back to the tent? Justify your answer.



Where along the river should you stop?

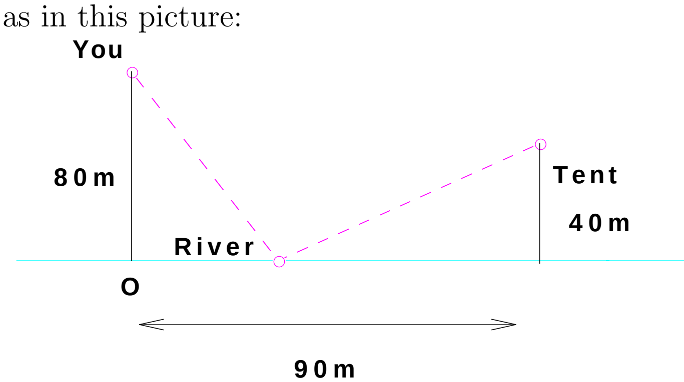
---

60. Make a sketch of the burning tent problem, with a moveable point R on the river. Describe how the value $YR+RT$ changes as R moves along the river.

61. Sketch a Cartesian graph for the values of $YR + RT$ as R moves along the full section of the river below Y and T.

Just label the axes "distance along the river" and "total distance for $YR + RT$." Exact numerical values aren't crucial as long as you have the general shape of the graph.

62. Suppose the dimensions in the burning tent scenario are as in this picture:



Where along the river should you stop?

Coordinatize the situation and express $YR+RT$ as a function of the distance OR. Sketch the graph of this function on your coordinate axes.

Suggestion: Make O the origin and $\overleftrightarrow{OR}$ the horizontal axis.

63. Pick a point R on the river that you're sure is *not* the best spot. How many other points P on the river have the same total distance associated with them (so $YR + RT = YP + PT$)?

The reflection technique could also be used to find a spot that's not *best.*

Write and Reflect:

64. One student investigating this problem told another student, "For each point along the line that's *not* minimal there's exactly one other point along the line of the river that's equally as bad." Explain how problem 61 might help students clarify this. Use, or model, the kind of language you as a teacher would like to hear them use.

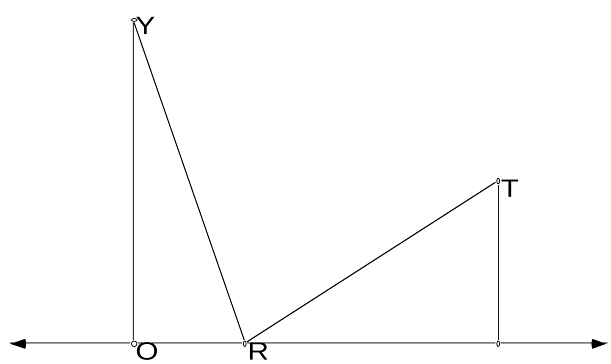
65. Another student claimed that the graph is a parabola. Is it? Why?

Ways to think about it:

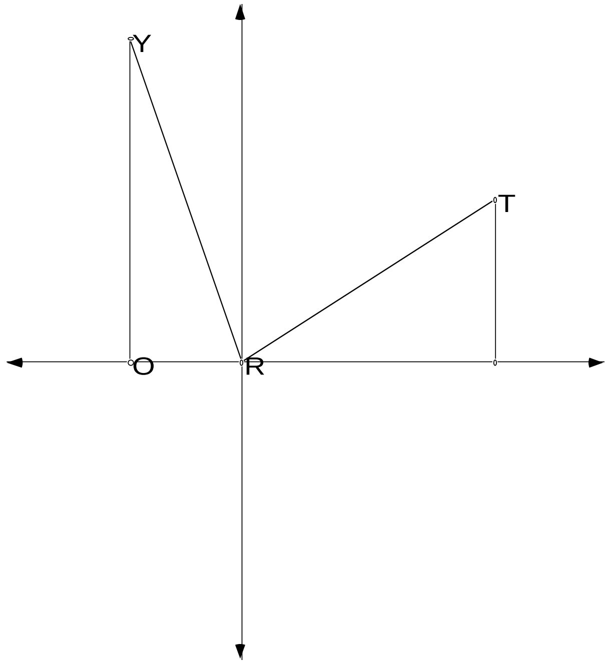
Setting up an an algebraic description of the "sum of distances" function in problem 62 requires skills that many students don't meet until the end of high school. But the use of dynamic geometry software allows students to set up the graph without any algebraic formalism. Here's how:

Set up a scale model of the situation in your software:

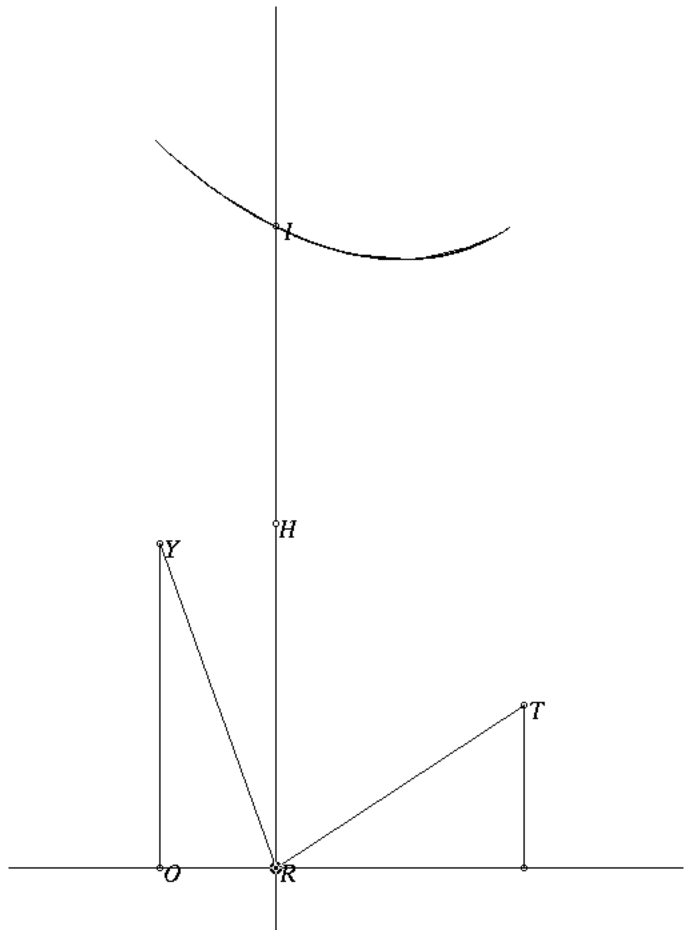
Making measurements in centimeters allows you to fit things on the screen to scale.



From R, raise up a perpendicular to the river:



On this perpendicular, mark off H and I so that $RH = RY$ and $RT = HI$. Put a trace on I:

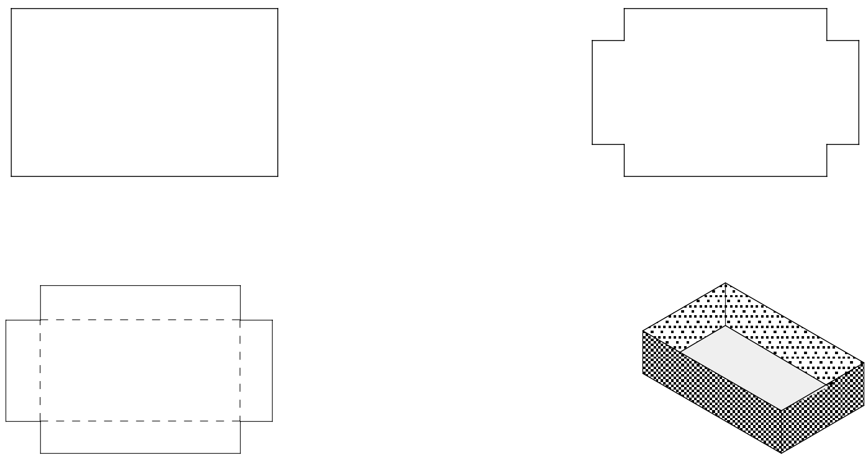


Now drag R back and forth along the shore, and the graph develops dynamically. This setup can also be used to investigate how the minimum value behaves as you change the distance from you to the shore, from the tent to the shore, or the length of the "run" along the shore.

1. The Box Problem

For several decades now, one of the most popular optimization problems in school mathematics has been the “box problem.” One version goes like this:

> **The Box Problem**: By cutting identical squares out of each corner of a $5'' \times 8''$ index card, you create a shape that can be folded into an open box.



> What size cut-out will maximize the volume of the box?

1. Solve the problem any way you can and, whether you come up with an exact or an approximate solution, pay attention to the process and methods you use in solving it. Record some of your thinking about how to solve the problem, and any insights you gained by thinking about the process/methods you used. Make sure to include the different approaches you tried, and which directions or methods failed to help, which seemed most helpful, and why.

Questions to Ask: *What are the various kinds of box shapes that you can create by cutting out different size corners? How can you tell if one box has more volume than another? Which (unmarked) dimensions in the diagram on the left determine the volume of the box?*

Ways to think about it:

As usual, the importance of a mathematical problem lies in the *thinking* students develop when working on it. So while working on the following problems we'll also focus on the particular methods being used to solve the box problem and the mathematical habits of mind behind those particular methods. These are the tools that students need in order to use, understand, and even create mathematics. Some of the mathematical habits that can be developed using different approaches to the box problem are: experimenting and tinkering with discrete data; reasoning by continuity (reasoning based on continuously varying data); visualization; graphical analysis; and algebraic thinking.

They'll need to know the formula for the volume of a box. How could you help them see that*? What other questions about volume or about the data might you ask them, or lead them to ask themselves?*
Some useful mathematical habits to develop:

Trends in discrete data. One of the nice things about the box problem is that you can get started without much machinery at all. For example, students can take an actual index card, cut out various size squares, measure the resulting volume, and, by carefully looking at the data, make some predictions. Answer the following questions in the context of a paper and scissors experiment around the box problem.

Look at extreme cases...

2. What is the smallest size cut-out you can make with a 5×8 rectangle? What is the largest?

Collect some data...

3. Gather some data for the box problem, make a table, and use your table to approximate the cut-out size that maximizes the volume.

Generalize...

4. What is the smallest size cut-out you can make with an $a \times b$ rectangle? What is the largest?

5. True or false: The size of the cut-out that maximizes the volume is half way between the numbers you figured out in problem 2. Explain.

6. Pick some volume that is less than the maximum one. How many differently sized cutouts produce that volume? Explain.

7. Some teachers have students approximate the maximum box by having students construct various boxes, fill them up with peanuts, sand, or something else and weigh the results. Critique this approach: would you find this appropriate in certain circumstances or not? If so, when?

There are many languages that can be used to describe functions, but the language of algebraic notation is still the most powerful and useful.

Analysis of the graph of a function. A graph of the function that expresses the volume of the box in terms of the sidelength of the cut-out would give you more information about the optimum cut-out (as well as all the other cut-outs) by analyzing the "geometry" of the graph. It's much easier to make this analysis if you have an algebraic expression that expresses the volume of the box in terms of the size of the cut-out. Soooo . . . .

You'll want to graph this function. If you plan to do this by calculator, make sure you choose a variable that is supported by your machine.

8. Choose a variable for the size of the cut-out and express the volume of the box as a function f of the cut-out size.

9. Compare the graph of the function f to the graph you obtained in problem 14. On what intervals do the graphs agree? Explain discrepancies on the intervals where they differ. Which graph is a more faithful model of the physical situation?

We're still working with a 5×8 rectangle here.

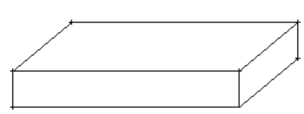
10. Use your graph to find, to three place accuracy,
(a) the size of the cut-out that maximizes the volume.
(b) the size of the cut-out that minimizes the volume.
(c) the size of the cut-out that produces a volume half way between the maximum and minimum volume.

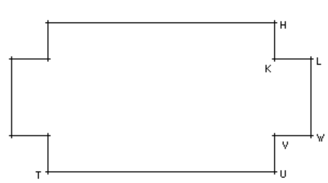
11. What is the "natural" domain for your function f? What values produce negative outputs from f? How can you explain these "negative volumes"?

12. Describe a graphing calculator algorithm that allows you to get a solution to the box problem accurate to n places.

13. Are there *any* starting rectangles that have the property that the optimum volume occurs half way between the two cut-outs that make the volume 0? Experiment with some graphs and make some conjectures. What mathematics would you use to prove your conjectures?

Continuous variation. The paper and scissors approach to the box problem captures some good thinking, but it also misses something crucial: the essentially *analytic* nature of this problem. We would all like students to look at the figure below and imagine HK starting at 0 and gradually growing to half of HU, imagining at the same time a continuum of boxes, starting out as a flat plate and gaining height, evolving into a vertical plate.



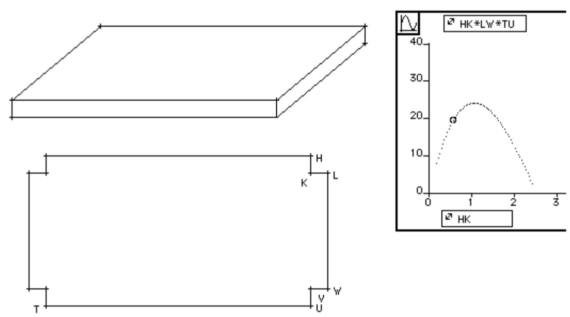


14. Use your dynamic geometry software to set up an experiment like the one described above. Important features of the design are:

(a) You should be able to use the mouse on the cut-out rectangle, changing its size by dragging on, say, K.

(b) As you change the size of the cut-out, the three-dimensional rendering of the box should change dynamically.

Even better, we'd like students to imagine a graph of the volume of the box against the size of the cutout evolving *along with* the growing and shrinking of HK. Numerical precision isn't important here. What's important is that students imagine a *continuum* of boxes, driven by H.



15. Add to your dynamic geometry sketch as follows:

(a) As you change the size of the cut-out, the graph should develop dynamically.

(b) The volume calculation (measured along the Y-axis), should be a product of three lengths in the diagram.

Write and Reflect:

16. Describe your thinking as you designed the dynamic geometry experiment in problem 14. What mathematics did you have to learn? What "hidden" geometry did you use?

17. What would a student need to know in order to set up this sketch? In what grade would you place the activity of designing the sketch? Of *using* the sketch?

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18. What is the smallest size cut-out you can make with an $a \times b$ rectangle? What is the largest? Explain, using your dynamic geometry model.

19. Use your dynamic geometry model to approximate the solution of the box problem.

20. What affects the precision of your approximation to the solution of the box problem if you use your dynamic geometry model?

21. True or false: The size of the cut-out that maximizes the volume is half way between the numbers you figured out in problem 18. Explain, using your dynamic geometry model.

22. Pick some volume that is less than the maximum one. How many size cutouts produce that volume? Explain, using your dynamic geometry model.

23. Use your dynamic geometry model to explain why there is only one cut-out that maximizes the volume.

24. If you drag the size of the cut-out "too far" (so that the rectangle doubles back on itself), the box and the graph may continue to change. Explain this behavior.

"Compare and contrast:" The two staples of English class.

Write and Reflect:

25. Compare and contrast the thinking you have used in each of the approaches so far (paper-and-scissors, graphical, and dynamic geometry).

A generic solution. Suppose you wanted to see how the optimum volume (or the cut-out size that produces it) varies as you change the size of the starting rectangle. It would be very helpful to have another function at hand: the one that expresses the optimum cutout as a function of the two dimensions of the starting rectangle.

26. Suppose the starting rectangle measures $a \times b$. If x is the size of the cut-out, express the volume of the box as a function g of three variables: x, a, and b.

27. Keeping a and b constant, use calculus to find the value of x that maximizes the volume of the box (in terms of a and b).

28. Use the results of problem 27 to check the approximate results that you've found so far to 5×8 box problem.

Thinking about students' thinking. Students often have the misconception that the size of the cutout does not affect the volume of the box, only the shape of it. What kinds of questions could you ask to help these students?

For Discussion:

As a class, outline some approaches to the box problem that would be appropriate for grades

The NCTM Standards *discusses the box problem and its use in middle school.*

- 7-9 prealgebra students
- 9-11 algebra students
- 9-11 geometry students
- 10-12 precalculus students
- 11-13 calculus students

For each grade level, write up a box problem activity that would draw on the mathematics students know.

---

29. Classify each of the solutions you would expect from students into one of the following types:
(a) it produces an approximate solution
(b) it produces an algorithm that could be used to approximate the solution to any desired degree of precision
(c) it produces an exact solution

(d) it gives a general method that could be used to obtain the exact solution to the box problem for *any* starting rectangle

30. If the rectangle that is used to make the box is a *square*, show how you can use algebra to obtain an exact solution to the box problem.

Take it Further. In problem 26, you set up a function of three variables, x, a, and b. There are several abstractions we can make from this that shed light on the original box problem.

31. Think of the value of x that maximizes the volume of the box as a function of the two variables a and b. Call this function G so that

$$G(a,b) = \text{the size of the cut-out that maximizes the volume of a box built from an } a \times b \text{ rectangle}$$

(a) Find an algebraic expression for $G(a,b)$.
(b) Make a table for G.

A spreadsheet is ideal for this.

$a \rightarrow$ $b \downarrow$	1	2	3	4	5	...
1						...
2						...
3						...
4						...
5						...
6						...
7						...
8						...
⋮						...

(c) Make a surface plot for G.

Mathematica *is an ideal tool for this.*

32. Are there *any* starting rectangles that have the property that the optimum volume occurs half way between the two cut-outs that make the volume 0? Prove your conjectures.

Another way to think of this situation is that each starting rectangle produces a volume *function*. In other words, for each a and b, we have a function $g_{(a,b)}$ which can then be graphed, differentiated, or transformed in some other way. Let's call this "higher order" function H:

$$H(a, b) = g_{(a,b)}$$

Whoa. Where did that *come from?*

where

$$g_{(a,b)}(x) = x(b - 2x)(a - 2x)$$

H can be thought of as a function that produces a "family" of functions. It is a mathematical way to capture the "collection of collections of boxes."

33. On the same set of axes, draw the graphs of

$$H(5, b), (1 \leq b \leq 10)$$

Describe any patterns that you see.

Unlike H, I returns a number.

34. Let $I(a, b)$ be the maximum volume of the box described by $H(a, b)$. Make a table of

$$I(5, b), (1 \leq b \leq 10)$$

and describe any patterns you see.

The optimal cut-out size will be rational if the derivative of the volume function has rational roots. The volume function is a cubic, so its derivative is a quadratic, and a quadratic has rational roots if its discriminant is a perfect square. See the book Complex Numbers and Arithmetic *in this series for ideas about what to do from here.*

35. Calculus teachers like to invent problems that come out nice. Give a relationship between a and b that guarantees that the value of the size of the cut-out that maximizes the box described by $H(a, b)$ is a *rational number*.

Note to field tests and reviewers

This section will continue in this spirit, analyzing several classical problems from calculus (the "soup can" problem and the like) that can be investigated and solved approximately using functions and graphs. We'll then conclude with a section that discusses the need for more powerful techniques if one wants to move beyond numerical approximations to specific problems (to solve generic optimization problems that involve several parameters, for example).

We'll also develop some graphical techniques that produce exact solutions (quadratic functions).

