Chapter III

Linearity and
Proportional Reasoning

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1. Mix it up

This chapter is about addition, multiplication, and combinations of those operations. These concepts start to develop in elementary school and continue into middle school. The study of the properties of those operations and combinations, though, is important from elementary school through higher mathematics. In this module, you'll explore those properties and the connections between early and more advanced mathematics.

The mixture blues

The following situation is one that can be—and has been—given to middle school students (and younger).

Product developers in the research lab for Whodunnit Jeans are trying to decide on a shade of blue for a new line of jeans. They have been mixing blue dye and clear water together, trying to find just the right shade.

In each of the problems below are two sets of blue-clear beaker combinations to mix. Each rectangle represents one beaker, and all beakers have the same volume of liquid in them. Decide which set is bluer, and explain how you made your decision. Find a way to decide without finding common denominators of two fractions!

1. \[A = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \quad B = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \]

Make believe that black is blue in these problems.

2. \[A = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \quad B = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \]

Notice how ambiguous the question “Which mixture is bluer?” becomes.

3. \[A = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \quad B = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \]

In other words, which mixture has more blue in it?

4. \[A = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \quad B = \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} \]
Reflect and Discuss

Three of the developers used different methods to make their predictions.

Nancy: I figured how many blue parts were used for each clear. If mixture A uses more blue for each clear than mixture B, then mixture A will be bluer. For example, in problem 1, mixture A uses 1 blue for 1 clear. Mixture B uses 1 blue for 3 clear, or $\frac{1}{3}$ blue for 1 clear. So Mixture A is bluer.

Terry: I calculated the number of blue minus the number of clear. The mixture with a greater difference is bluer. For example, in problem 1, mixture A’s difference is 0 and mixture B’s difference is $-2$. So mixture A is bluer.

Sid: I looked at how much blue there is compared to the total amount of liquid. For example, in problem 1, mixture A has 1 blue out of 2 beakers while mixture B has 1 blue out of 4 beakers. Mixture A is bluer.

5. Do two of the developers always agree with each other?
6. Some students may have trouble accepting that one of these methods is incorrect. Find an extreme example: two mixtures for which the methods don’t all agree, but it’s very clear which is incorrect.

Nancy decided she preferred this mixture over all the others:

If possible, connect each problem with a particular developer.

Sid wanted to make a big vat the same shade as Nancy’s mixture.

7. If Sid used 45 beakers filled with blue dye, how much water (clear) was needed to get the same shade as Nancy’s mixture?
8. If the total amount of liquid in the vat was 1000 units, how much dye and how much water were needed?
9. How is the reasoning needed to answer each of problems 7 and 8 related to the developers’ methods described above?

Reflect and Discuss

As you might know, proportional reasoning was required to answer the problems in this section. This may have been more evident in problems 7–9, but even in problems 1–4,
that reasoning was helpful. Sid and Nancy compared quantities as ratios or rates. However, they used the parts and the whole of the mixture in different ways.

10. How might making the distinctions between the two methods (part-to-part ratio or part-to-whole ratio) help students in their understanding of proportional reasoning?

**Putting it together**

Proportional reasoning involves multiplication: When one quantity is multiplied by a factor, the other quantity must also be multiplied by that factor to maintain the proportion. Although this reasoning is developed extensively throughout middle school and high school, many students fail to connect the different ways this reasoning is used, especially when small twists are introduced.

The developers at Whodunnit jeans decided to see what would happen if they took two different mixtures and mixed them together. They called this the “union” of the two mixtures. For example, they took the two mixtures $A$ and $B$ from problem 2:

$$A = \begin{cases} \text{blue} & \text{clear} \\ \text{blue} & \text{clear} \\ \text{blue} & \text{clear} \end{cases}$$
$$B = \begin{cases} \text{blue} & \text{clear} \\ \text{blue} & \text{clear} \\ \text{clear} & \text{clear} \end{cases}$$

and formed the union of the mixtures (they named it $A \cup B$):

$$A \cup B = \begin{cases} \text{blue} & \text{clear} & \text{blue} & \text{clear} & \text{blue} & \text{clear} & \text{clear} & \text{clear} \end{cases}$$

11. In this example, which is bluest: $A$, $B$, or $A \cup B$?

12. Can $A \cup B$ ever be bluer than both $A$ and $B$? If not, why? If so, when?

13. Given two (possible different) mixtures $A$ and $B$, can $A \cup B$ ever be just as blue as one of $A$ or $B$? If not, why? If so, when?

14. Suppose you could combine several vats of the mixture $A$ above with one or more vats of the mixture $B$ above. (For example, three of mixture $A$ would have 15 total parts, 9 blue and 6 clear.) Could such a combination ever be bluer than mixture $A$? If not, why? If so, when?
Sid invented a term called the “blueness quotient” (BQ) for a mixture, which is the ratio of blue beakers to total beakers. In the mixture in problem 2, the BQ for $A$ is $\frac{3}{5}$, and the BQ for $B$ is $\frac{2}{4}$ or $\frac{1}{2}$.

15. Interpret the BQ as a percentage. What does this percentage mean for the mixture with a BQ of $\frac{3}{5}$?

16. Do the BQs of two mixtures tell you anything about the comparison of the two mixtures?

17. Nancy preferred a shade created by mixing 3 beakers of blue dye with 2 beakers of water, and she kept a large glass container of this mixture on her lab table. Terry filled a vat with that shade by using 60 beakers of blue dye and 40 beakers of water. For a (very bad) practical joke, Sid added 3 beakers of blue dye to Nancy’s container and another 3 beakers of blue dye to Terry’s vat. Do the mixtures still have the same shade? Explain.

18. Is it possible to give an algorithm for computing the BQ of $A \cup B$ if you know the BQs for $A$ and $B$? For example, suppose you combined a mixture with a BQ of $\frac{1}{2}$ with a mixture with a BQ of $\frac{1}{3}$. If so, how? If not, could you do it if you had more information?

19. Suppose you want to combine 7 ml of a mixture with a BQ of $A$ and 12 ml of a mixture with a BQ of $B$. Could you find the BQ of the result? If so, how? If not, why?

**Reflect and Discuss**

20. Why can’t you add fractions in the usual way (that is, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$) to get the BQ of the union? Does this strange fraction addition appear in any other mathematics topics for middle school and high school students?

**And mix some more**

Traditional algebra textbooks, particularly those published before the National Council of Teachers of Mathematics released its *Curriculum and Evaluation Standards for School Mathematics* in 1989, often included “word” or “story” problems. In contrast to the regular exercises, which were generally computational or procedural in nature, these word problems required students to apply their computations and procedures to problems presented using a context.
Often, the contexts became a way to classify the problems: coin problems, train problems, and mixture problems were only a few of the types. Students learned to do the problems by rote: They recognized the type of problem and then inserted the numbers into equations associated with the type. Years later, most would probably say, "I don't remember how to do mixture problems."

Students would be better served if they could reason about the problems, much in the same way you reasoned through the problems in this session.

Here is one example of a classic mixture problem (sometimes called a "solutions" problem):

Five ounces of a 30% alcohol solution is mixed with ten ounces of a 50% alcohol solution. What is the percent of alcohol in the result?

21. Which of the blue-dye problems in this session does this seem to be the most like? Write a new blue-dye problem that uses the amounts in this solutions problem and would give the same result.

22. A related, but more difficult, problem might ask how much of each solution (30% and 50%) should be used to create 100 ounces of a 45% solution. Rewrite this new problem to be more like a blue-dye problem. (Algebra would probably be needed to actually solve this problem.)

Other contexts can be solved in the same way. For each of these, write a corresponding dye-mixture problem that gives the same result.

23. **Speed** A traveler has a 60-mile journey ahead. If he drives half the distance at a speed of 30 miles per hour and the rest of the way at a speed of 60 miles per hour, what is the average speed for the trip?

24. **Weighted average** Ms. Callahan calculates final grades in her class using five tests and an exam, which counts as three tests. Lon averaged 83 for the five tests and earned a 78 on the exam. What is his final grade?

25. **Mixed nuts** Peanuts cost $3 a pound, while cashews cost $7 a pound. If you mix 2.5 pounds of peanuts with 0.7 pounds of cashews, how much should the mixture cost per pound?
Ways to think about it

For your convenience, the problems are restated in the margin.

5. The three methods do not give identical numerical results, but two use basically the same type of reasoning.

6. Extreme examples might include a mixture that’s all clear, or all blue. Other examples might be mixtures that have a large number of one type of liquid but only one of the other type.

7. If Sid made several copies of Nancy’s mixture, how many blue beakers would be included in two copies? Three? Four? How many copies would give 45 blue beakers?

8. Try a similar tactic to what you did for problem 7, but focus on the total number of beakers needed.

9. Each of the two problems requires the same focus as one of the two correct methods.

12–13. You might try looking at examples. Include extreme examples, but also try examples for which $A$ and $B$ are nearly (or exactly) identical.

15. The BQ is a fraction or ratio. The fraction describes what part of what whole?

16. Each BQ represents a fraction with a concrete meaning. When you compare those two fractions, what does that tell you?

17. Compare the BQs of the two resulting mixtures.

18. Be sure to test any algorithms you create, using a variety of mixtures. Be wary of making assumptions about the mixture based on the BQ. Look back at problem 17 and use the mixtures given there.
If you find you are making assumptions, what assumption are you making? Can you turn that assumption into a question to ask about the mixtures?

If you think you’re not making assumptions and you know whether an algorithm is possible, consider what extra information about a mixture might be lost when calculating the BQ. Could that extra information help you create an algorithm or make the one you have more efficient?

19. Note that the actual volume of a beaker of liquid has never been established. Can you assume a particular volume? Could you assume different size beakers for each mixture? You might want to try this with examples for the two mixtures.

20. Recall what information is lost when you calculate the BQ of some mixtures.

21. It may help to rewrite the percentages as ratios. For example, 30% alcohol is how many parts alcohol to how many parts water? Draw a parallel between that and a blue-dye mixture.

22–25. For each of these problems, identify what kind of ratios are being used, and how they are being combined. Draw parallels between the situations and the blue-dye problems.

**Problem:** Suppose you want to combine 7 ml of a mixture with a BQ of A and 12 ml of a mixture with a BQ of B. Could you find the BQ of the result? If so, how? If not, why?

**Problem:** Why can’t you add fractions in the usual way to get the BQ of the union?

**Problem:** Write a new blue-dye problem that uses the amounts in this solutions problem and would give the same result.
2. Filling in the gaps

In section 1, you looked at situations which required proportional reasoning. This type of reasoning is important in many different contexts. In this session, you will explore how proportional reasoning can be helpful even in situations for which this type of reasoning doesn’t necessarily lead to exact solutions.

One of the main themes of this section is interpolation: filling in values of a function between two known values. Before cheap pocket calculators were available, people used tables to find values of functions such as logarithms and trigonometric functions. Often, the value wanted would not be listed on the table, so the person would interpolate from the surrounding values. For example, the value of \( \sin 45.7 \) could be interpolated from a table that gave values for \( \sin 45 \) and \( \sin 46 \).

Reflect and Discuss

1. A common type of elementary math problem is to find the next term in a sequence such as 1, 2, 4, \ldots. Discuss possible answers for this problem and the reasoning used.

One common answer to problem 1 is 8; the term after 8 would be 16, then 32 \ldots. Keeping this particular pattern in mind, change the problem.

2. Instead of continuing the sequence, fill in more terms between the existing ones. That is, if the sequence were 1, _, 2, _, 4, \ldots, what could go in the blanks?

There are at least two reasonable answers for problem 2. One involves thinking of the sequence as additive; to get from one term to the next, you have to add something each time (but not necessarily always the same something). Another answer involves thinking of it as multiplicative; to get from one term to the next, you have to multiply something each time.

3. Find both an additive pattern and a multiplicative pattern that will generate a sequence 1, _, 2, _, 4, \ldots.

4. In a way, you can think of the numbers you put in each blank as being “halfway” between the terms on either side. This gives at least two meanings for “halfway.” Use similar processes (adding and multiplying) to find two different numbers that are (sort of) \( \frac{1}{3} \) of the way between 1 and 2 in the sequence. Then use those answers to find two numbers \( \frac{2}{3} \) of the way between.
5. Can you find two numbers any (rational) fraction of the way between 1 and 2 in the sequence?

6. Answer problems 4 and 5 for numbers between 2 and 4 in the sequence.

Reflect and Discuss

7. How did you find your answers to problems 2–6?

Now consider a graphical interpretation of finding numbers a fractional part of the way between the numbers in the sequence. Call the term number \( n \) and the corresponding term \( a_n \). So, for example, if \( n = 1 \), then \( a_n = a_1 = 1 \). Similarly, if \( n = 3 \), then \( a_n = a_3 = 4 \).

8. Below is a plot of the first three terms of the doubling sequence. The term number or index, \( n \), is on the horizontal axis and the sequence value, \( a_n \), is on the vertical axis. Connect the points using straight lines. That is, connect (1, 1) to (2, 2) and (2, 2) to (3, 4).

Use a straightedge if you have one available.
9. Mark points on the lines you just drew for which \( n = 1 \frac{1}{3}, \ n = 2 \frac{1}{3}, \) and \( n = 2 \frac{3}{4}. \) Those points represent “between” sequence values. Use those points to find the corresponding values of \( a. \)

One algebraic definition of the doubling sequence is \( a_n = 2^{n-1}. \) If you graph \( y = 2^{n-1}, \) perhaps on a graphing calculator, you get a nice, smooth curve through the points.

10. Here are actual values and approximations, accurate to the thousandth place, for the points on the curve corresponding to the \( n \) values in problem 9:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1 ( \frac{1}{3} )</th>
<th>2 ( \frac{1}{3} )</th>
<th>2 ( \frac{3}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n, ) exact</td>
<td>( \sqrt{2} )</td>
<td>( 2 \sqrt{2} )</td>
<td>( 2 \sqrt{8} )</td>
</tr>
<tr>
<td>( a_n, ) appr.</td>
<td>1.260</td>
<td>2.520</td>
<td>3.364</td>
</tr>
</tbody>
</table>

(a) How close were your approximations in problem 9?
(b) Explain why it makes sense that some approximations are closer than others.

11. Following is another plot of the first three terms in the sequence, along with the actual “halfway” points that lie on the curve. (Labels give approximate values, to the nearest thousandth.)
(a) Once again, draw lines on the graph and find approximate values for $a_n$ when $n$ is $1\frac{1}{3}$, $2\frac{1}{3}$, and $2\frac{3}{4}$.

(b) You probably would expect that your new approximations are closer than the old ones. Explain why that would be true.

To move from one point to another on any graph, you have to move left or right by a certain amount (call it $\Delta n$) and also up or down by a certain amount (call it $\Delta a$). Lines, like the ones you created by connecting points in problem 8, have a special characteristic: Every time you move that $\Delta n$ distance, the distance you move up or down is always the same: $\Delta a$.

This characteristic of lines allows you to work with them very easily. Think again about approximating a point $\frac{1}{3}$ of the way from $(1,1)$ to $(2,2)$, using the line you drew in problem 8. To move from $n = 1$ to $n = 1\frac{1}{3}$, you move $\Delta n = \frac{1}{3}$ unit to the right. Three of these jumps—all the same size—will land you on $(2,2)$.

12. Since you’re moving along a line, each jump up is the same size (in both the $n$ and the $a$ directions). If you make three equal-sized vertical jumps from $a = 1$ to $a = 2$, what is the length of each jump? Where are you when $n = 1\frac{1}{3}$?

13. The following questions use the same reasoning to help you find a numerical way to make the other approximations you

The uppercase Greek letter delta ($\Delta$) is often used in mathematics to denote a difference. For example, $\Delta n$ means the difference between two $n$ values: $n_j - n_i$.

Graphs that aren’t lines don’t have this characteristic. For example, when you move from $(1,1)$ to $(2,2)$ on the doubling graph, $\Delta n$ is 1 and $\Delta a$ is 1. When you move from $(2,2)$ to $(3,4)$, $\Delta n$ is still 1, but $\Delta a$ is now 2. The distances you move depend on where you are on the graph.
made graphically in problem 9. Work through the steps to approximate the value for \( a_n \) when \( n = 2 \frac{3}{4} \).

(a) First consider what fraction of the way from one point to the next you have to jump horizontally. For example, to get to \( 1 \frac{1}{3} \), you had to go \( \frac{1}{3} \) of the way from 1 to 2.

(b) When you jump horizontally, you also have to jump vertically. What fraction of the way between two points do you have to jump? How far is that? >

(c) Now figure out where you land when you make that jump. Don’t forget, you started at a point other than the origin.

14. Go through the steps of problem 13 and generalize the method to give an approximation for \( a_n \) when \( m < n < m + 1 \) and you know \( a_m \) and \( a_{m+1} \).

15. Now, generalize the method further, to give an approximation for any \( a_n \) when \( m < n < p \) and you know \( a_m \) and \( a_p \). Note that \( m \) and \( p \) may be any numbers, including nonintegers.

16. The slope of a line with independent variable \( x \) (on the horizontal axis) and dependent variable \( y \) (on the vertical axis) can be defined as \( \frac{\Delta y}{\Delta x} \). Does the slope of the line between \( (m, a_m) \) and \( (p, a_p) \) appear in your generalization?

In section 1, you worked with mixture or weighted average problems like the following:

The BQ (blueness quotient) of a mixture is the ratio of the number of blue dye beakers to the total number of beakers in a mixture. Suppose you combine \( x \) ml of a mixture with a BQ of \( A \) with \( y \) ml of a mixture with a BQ of \( B \).

17. What is the BQ of the resulting combination?

**Reflect and Discuss**

18. How is the blue-dye problem above related to the interpolation problem?

The weighted average you used to find the BQ of the result is a linear combination of the BQs \( A \) and \( B \). A **linear combination** of two quantities is the sum when each is multiplied by a number or variable. For example, \( 3x + 2y \) can be considered a linear combination of \( x \) and \( y \). It can also be considered a linear combination of 3 and 2.
19. Use a linear combination for the interpolation problem. That is, find a way to write a value \( a_n \) as a linear combination of \( a_m \) and \( a_p \), where \( m < n < p \):

\[
a_n \approx ____a_m + ____a_p
\]

**Local linearity**

As you saw in problem 11, having more data points (or at least, data closer together) leads to better approximations. A similar concept is often used by mathematicians working with nonlinear functions.

Using a graphing calculator, graph a polynomial or exponential function. For example, you might graph the doubling function, \( y = 2^x \), or a polynomial such as \( y = 3x^4 - 2x^3 + x - 1 \). Try to get a function that looks “curvy.”

- Find a section of the graph that looks particularly curvy—as far from a line as the graph seems to get. Zoom in on that section.
- Now find a section in that window that looks curvy and zoom in again.
- Repeat this for several zooms.

**Reflect and Discuss**

20. Did you find it more and more difficult to find a particularly curvy part of the graph? What did your final zooms look like?

The phenomenon you probably observed is what mathematicians call **local linearity**. If you look at the graph in a small enough “neighborhood” around a point, in most cases the graph will appear to be a line.

A line has a slope, because for a given horizontal change, the vertical change is always constant. Because a nonlinear curve can have different vertical changes depending on where you look, it can’t have a single slope. You can, however, find the slope of a curve at a particular point.

21. The slope of a curve at a point can be thought of as the slope of the line tangent to the curve, at that point. A **secant line** is a line through two points on the curve. Here is a graph that includes the curve \( y = x^3 \), the tangent line...
at a point $x$, and the secant through two points at $x$ and $x+h$.

(a) What happens to the secant line as $h$ gets smaller and eventually becomes 0?
(b) Find an expression for the slope of this secant line. What happens to the expression as $h$ gets smaller (approaches 0)?

22. In mathematics, taking a variable to a certain value is called taking a limit. For example, the limit of $x+h$ as $h$ goes to 0 is simply $x$ (provided $x$ and $h$ are independent):

$$\lim_{h \to 0} x + h = x$$

(a) Write an expression for the slope of the secant line for a general function $f(x)$, through the points $(x, f(x))$ and $(x+h, f(x+h))$.
(b) The slope of the tangent line is the limit of the secant line’s slope, as $h$ goes to 0. Use the limit notation to write an expression for the slope of the tangent line.

Congratulations! You have not only found the derivative of $x^3$ in problem 21, you wrote the general definition of a derivative (problem 22). Derivatives are one of the fundamental concepts in calculus.
Ways to think about it

1. There are actually an infinite number of possible next terms for this sequence. The most obvious can be found by looking for patterns in ratios (\(\frac{\text{one term}}{\text{the previous term}}\)) and differences (\(\text{one term minus the previous term}\)).

2. You probably found one of these in your answer to problem 2, in which case your first step is to figure out if you used a multiplicative pattern or an additive one.

   If you need to find an additive pattern, first figure out an additive pattern for the “parent” sequence, 1, 2, 4, 8, \ldots. Now you have to make additions that give the same sequence with one additional value between each pair in the parent sequence. Remember, you don’t have to use a different value each time, as long as you make an identifiable pattern.

   If you need to find a multiplicative pattern, consider this: In the parent sequence, you multiply by 2 to get the next term. If the sequence were 1, \(\ldots\), 2, 4, \ldots, what would you multiply each time? Suppose you multiply by \(r\). The first term is 1; the second is \(r\). What is the third term, using \(r\)? For that to be 2, what does \(r\) have to be?

3. Use the same idea from problem 2, above. To include numbers \(\frac{1}{3}\) and \(\frac{2}{3}\) of the way between the original terms, the new sequence would have to look like 1, \(\ldots\), 2, \(\ldots\), 4, \(\ldots\).

10. To explain why some approximations are closer than others, it might help to make a rough sketch of what the actual curve through the points would look like. (The curve is not a series of lines.) Where is that curve furthest from the lines?

14. Look back at your work in problem 13. For \(n = 2\frac{1}{3}\), you should have used \(m = 2\) and \(m + 1 = 3\), with corresponding sequence values \(a_m = 2\) and \(a_{m+1} = 4\).

15. Again, look back at your work in problem 13. The big new thing to remember is that the distance from \(m\) to \(p\) is not necessarily 1 unit, so jumping \(\frac{1}{3}\) of the way from \(m\) to \(p\) is not jumping \(\frac{1}{3}\) unit. (For example, how far do you jump if you jump \(\frac{1}{3}\) of the way from 2 to 5?)

Problem: Discuss possible next terms for the sequence 1, 2, 4, \ldots and the reasoning used.

Problem: Find both an additive pattern and a multiplicative pattern that will generate a sequence 1, \(\ldots\), 2, \(\ldots\), 4, \(\ldots\).

Problem: Find two different numbers that are (sort of) \(\frac{1}{3}\) of the way between 1 and 2 in the sequence; find two that are \(\frac{2}{3}\) of the way between.

Problem: How close were your approximations to the actual values? Explain why it makes sense that some approximations are closer than others.

Problem: Generalize the method to give an approximation for any \(a_n\) when \(m < n < m + 1\) and you know \(a_m\) and \(a_{m+1}\).

Problem: Generalize the method further to give an approximation for any \(a_n\) when \(m < n < p\) and you know \(a_m\) and \(a_p\).
16. Look for differences \( a_p - a_m \) and \( p - m \) in your generalization. If you find them but they don’t appear as a ratio or fraction, is it possible to create that ratio? Would doing so make your generalization look simpler or more familiar in some way?

17. Answer the following questions, then use that information to find the BQ: How many milliliters of blue dye does each mixture contribute? What is the total amount of liquid in the combination?

19. You may already have written this in your generalization from problem 15. If not, see if you can use algebra to rewrite your generalization in the form of a linear combination of \( a_m \) and \( a_p \).

21. If you’re having trouble visualizing this, place a straightedge (like a ruler or the side of a piece of paper) on the illustration, where the secant line is now. If \( h \) gets a little smaller, how does this affect \( x \) and \( x + h \)? Where would the points \( (x, x^3) \) and \( (x + h, (x + h)^3) \) be? Move the straightedge to represent the new secant line . . .
3. Guess my rule

In section 1, you looked at situations for which proportional reasoning and relative comparisons were important. In section 2, you looked at interpolation and how lines can be used to approximate curves.

This section explores another way of defining both linear and nonlinear functions: by identities and properties of functions. The problems have a distinctly algebraic flavor, but you may also want to try representing the functions by tables and graphs.

To begin, consider these four symbols: \{\square, \triangle, \blacklozenge, \blacktriangle\}. You can “add” two symbols by superimposing one on top of the other, giving this addition table:

\[
\begin{array}{c|cccc}
+ & \square & \triangle & \blacklozenge & \blacktriangle \\
\hline
\square & \square & \triangle & \blacklozenge & \blacktriangle \\
\triangle & \triangle & \blacklozenge & \blacktriangle & \square \\
\blacklozenge & \blacklozenge & \blacktriangle & \square & \triangle \\
\blacktriangle & \blacktriangle & \square & \triangle & \blacklozenge \\
\end{array}
\]

1. Suppose you add two symbols and then rotate the result 180°. (For example, add \square to \triangle. The result is \blacklozenge, which becomes \blacktriangle when rotated 180°.) Then, with the same original symbols, change the order: first rotate (each \blacklozenge becomes \blacktriangle) and then add (\blacktriangle + \blacktriangle = \square).

For any choice of two symbols, will you always get the same final answer by adding and then rotating the result as you would if you first rotate and then add the results?

For functions and operations using real numbers, students often assume that they always can do these things in either order. How many algebra (and even calculus) students trip up by thinking \((x + y)^2 = x^2 + y^2\)?

Of course, adding two numbers then performing some rule on the result will not always give the same result as performing the rule on each number before adding. The next question to ask is, for what kind of rules will these have the same result? In this session, you will work to answer that question for real-valued functions.

Some college professors call this the “Freshman Dream.”
What kind of assumptions do you want to make about $F$?

For example, if $F(7) = 35$ and $F(9) = 45$, then $F(16)$ would be 80.

You’ll see more about why you can’t rely on patterns in problem 7.

**Problem**

**Mystery Function Problem**

A function, $F$, has only real numbers in its domain (its inputs), and its range (outputs) also consists of real numbers. For any real numbers $a$ and $b$, $F(a + b) = F(a) + F(b)$. That is, if you put in the sum of two numbers, what comes out is the sum of the outputs for the two numbers fed in separately. What does $F$ look like, algebraically?

2. Is enough information given in the Mystery Function Problem to determine the value of $F(0)$? Is enough information given to determine the value of $F(1)$?

3. Suppose you also know that $F(1) = -\frac{5}{6}$. Fill in the table below using only the information given. That is, don’t depend on finding patterns. ⇒ Be sure to fill in the “Reason” column!

<table>
<thead>
<tr>
<th>$x$</th>
<th>$F(x)$</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{5}{6}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>any positive integer $n$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>any integer $n$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Show that if $F(a + b) = F(a) + F(b)$ and $c$ is an integer, then $F(ca) = cF(a)$ (even if $a$ is not an integer).
5. You probably have some idea for a rule for $F$. Fill in the next table, again without using such a rule. Only use the clue that $F(a+b) = F(a) + F(b)$ and results from previous problems.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$F(x)$</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{n}$ where $n$ is an integer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{m}{n}$ where $m$ and $n$ are integers</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Show that if $F(a+b) = F(a) + F(b)$ and $r$ is a rational number, then $F(ra) = rF(a)$ (even if $a$ is irrational).

Reflect and Discuss

7. Can you determine a value for $F(\sqrt{2})$? For $F(\pi)$? Do problems 4 and 6 imply that $F(ka) = kF(a)$ for all real numbers $k$?

An identity is an equation that applies for all values of the variables involved. For example, if $a$ and $b$ are real, then $a^2 - b^2 = (a+b)(a-b)$ is an identity because it is true for all values of $a$ and $b$ that make sense in the equation. (In this equation, all real numbers $a$ and $b$ make sense. If the identity involved square roots, you might have to exclude negative numbers.) However, $x^2 - 3x + 2 = 0$ is not an identity, because it is true only for two values of $x$.

The equation $F(a+b) = F(a) + F(b)$ is sometimes an identity, depending on what the function $F$ is. A function for which $F(a+b) = F(a) + F(b)$ is an identity is called an additive homomorphism, or an addition-preserving function.

The idea here is that the domain (input set) of the function is a set (in this case, the real numbers) with an operation (in this case, addition). The range (output set) of the function is another set (also the real numbers) with another operation.
(also addition). The identity \( F(a + b) = F(a) + F(b) \) says that addition in the domain corresponds to addition in the range. Or, stated another way, if you add in the domain, then move to the range by \( F \), you get the same result as if you first move to the range, then add.

\[
\begin{array}{c}
\text{Domain} \\
\downarrow \\
F \\
\downarrow \\
\text{Range}
\end{array}
\quad
\begin{array}{c}
a \\
F \\
a + b \\
F \\
F(a + b) = F(a) + F(b)
\end{array}
\]

This identity, combined with the property that \( F(ka) = kF(a) \) for all possible values of \( k \), \( \geq \) is often used as the definition of a linear mapping.

**DEFINITION**

A linear mapping is a function \( F \) such that

- \( F(a + b) = F(a) + F(b) \)
- \( F(ka) = kF(a) \)

For a mapping from the real numbers into the real numbers, the first property guarantees the second property if the function is continuous—that is, as long as there are no “holes” or “breaks” in the graph.

8. Are these functions linear mappings? Why or why not?
   (a) \( f_1(x) = \) the cost of \( x \) pounds of beans at the health food store
       (Beans are sold in bulk; you can buy any amount you want.)
   (b) \( f_2(x) = \) the area of a square \( x \) meters on a side
   (c) \( f_3(x) = \) the total amount of money in your bank account after \( x \) years, if you start with \$1000 at year 0, and never withdraw or deposit money, and interest is compounded annually at a constant rate of \( r \)%
   (d) \( f_4(x, y) = \) the cost of \( x \) pounds of beans and \( y \) pounds of rice at the health food store

Linear mappings can appear with sets other than real numbers, as you’ll see in the next problem.
9. Is the described function a linear mapping? Why or why not?
   (a) For an angle measure \( r \), \( f_5(r) \) is a rotation \( r^\circ \) counterclockwise about point \( D \). Addition of rotations is their composition, that is, \( f_5(r) + f_5(s) \) is the rotation through \( r^\circ \) followed by the rotation through \( s^\circ \).
   (b) The function \( f_6 \) takes a point on a coordinate plane for its input and moves it 3 units to the left and 4 units up. Addition of two points \((a_1, b_1) \) and \((a_2, b_2) \) is defined as \( (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \).
   (c) For a mixture of beakers filled with either blue dye or water, the blueness quotient (BQ) is the ratio of the number of blue-dye beakers to the total number of beakers. The sum of two mixtures is their union. Is the BQ a linear mapping? That is, is the BQ of the union equal to the sum of the BQs?

Reflect and Discuss

Here’s a function that might surprise you: \( f_8(x) = \) the total cost of an order of \( x \) pounds of beans from beans.com, including a $4.95 shipping charge.

10. Is \( f_8 \) a linear mapping?

11. What do you think it means to ask, “Is a function \( g \) linear?”

Standard English words can mean different things in different contexts. Mathematical terms can mean different things as well. In the case of “linear,” there are two common senses which are closely related. One sense is that of high school algebra: a function is linear if its graph is a line. Function \( f_8 \) would then be considered a linear function.

The other sense is the definition of a linear mapping. This sense is used in fields such as linear algebra, which includes matrices. While the required properties exclude functions of the type \( y = mx + b \) where \( b \) is nonzero, those properties allow mathematicians to do more with a linear mapping. Knowing a function is a linear mapping is much more powerful than knowing a function is a linear function.

12. The following questions are based on a linear mapping \( T \) from \( \mathbb{R} \) to \( \mathbb{R} \), where \( T(1) = -4 \).
   (a) Find \( T(3) \).
   (b) Find \( a \) so that \( T(x) = ax \).
(c) Find all $x$ so that $T(x) = 3$.
(d) Which of problems 12a–12c could you still answer if you did not know that $T$ is a linear mapping? Why?

13. Here is a simplification of the MYSTERY FUNCTION PROBLEM: Describe all polynomials $P$, with real numbers for their domain and range, for which $P(a + b) = P(a) + P(b)$. Be sure to explain why you know you’ve described all of them. >

Linear combinations

14. Using a linear combination, the two requirements for linear mappings can be combined into one statement:

$$F \text{ is a linear mapping if and only if } F(ax + by) = aF(x) + bF(y).$$

Since this is an “if and only if” statement, both directions have to be proved.
(a) Show that if $F$ is a linear mapping, then $F(ax + by) = aF(x) + bF(y)$.
(b) Show that if $F(ax + by) = aF(x) + bF(y)$, then $F$ is a linear mapping.

15. In section 2, you found a formula for interpolating data between two points. Your formula probably looked something like this, but using subscripts instead of function notation:

$$a(n) = a(m) + \frac{n - m}{p - m} (a(p) - a(m))$$

Let $x = \frac{n - m}{p - m}$.
(a) Find an expression for $n$ that uses $x$, $m$, and $p$.
(b) Replace each $n$ in the formula above with the expression you just found. Rewrite the result using linear combinations on both sides of the formula. That is, write it in the form

$$a(\text{linear combination of } p \text{ and } m) = \text{linear combination of } p \text{ and } m$$

(c) What values should be used for $x$?
(d) Describe all polynomials for which the interpolation is exact, not an approximation. Explain why this makes sense, using both symbols and a graphical argument.
(e) Polynomials for which the interpolation is exact do not have to be linear mappings. What in the linear interpolation formula is different from the definition of linear mappings and allows this to be true?
Ways to think about it

1. You probably will find it easiest to show it works for different cases: if one symbol is □, if one symbol is □, and so on. For some of those cases, whether it works for every choice of the other symbol may be easy to show; for others, you might have to break it into even more specific cases.

2. The only information you have is that \( F(a + b) = F(a) + F(b) \). To find the value of \( F(0) \), it will have to appear in such an equation. Rewrite the equation so that \( F(0) \) appears somewhere, then solve for \( F(0) \).

   Would something similar work for \( F(1) \)? If so, do it. If not, explain why not. It might help to think of some examples of \( F \) where \( F(a + b) = F(a) + F(b) \) and compare \( F(1) \) in each case.

3. You should be able to fill in the value for \( F(0) \). You now have function values for two input numbers, 0 and 1. The equation in the Mystery Function problem statement refers to sums. Can you write sums, using the input values for which you have function values, to create other inputs? For the negative integers, recall or read the “ways to think about” problem 2 for finding \( F(0) \). Can you do something similar to find values for negative integers?

4. This proof will require more than simple algebraic manipulation, because you want to show it’s true for all integers \( c \). One way to approach the problem is to use mathematical induction. With mathematical induction, you show that something is true for a specific case, such as \( a = 1 \). Then you use the information you have to show that, assuming it’s true for cases up to, say, \( a = n \), then it must be true for \( a = n + 1 \). (Since you’ve shown it’s true for \( a = 1 \), then, it must be true for \( a = 2 \). But that means it’s true for \( a = 3 \), which in turn means ...) In this case, you would have to work in two directions. Going up will prove the statement is true for positive integers but not negative integers. A second (similar) argument will be needed to show the statement is true going down, as well. (That is, assuming it’s true for \( a = n \), show it’s also true for \( a = n - 1 \).

5. The table starts with \( x = \frac{1}{2} \). Again, you know the function values for integers, and you know how to find function values for inputs expressed as sums. How can you use sums with \( \frac{1}{2} \) to get an integer (and so a function value you know)? To find values for rational numbers that are not unit frac-
96 WAYS TO THINK ABOUT MATHEMATICS

Problem: Show that if $F(a + b) = F(a) + F(b)$ and $r$ is a rational number, then $F(ra) = rF(a)$.

Problem: Can you determine a value for $F(\sqrt{2})$? For $F(\pi)$? Do problems 4 and 6 imply that $F(ka) = kF(a)$ for all real numbers $k$?

Problem: Are these functions linear mappings? Why or why not?

Problem: Which of problems 12a–12c could you still answer if you did not know that $T$ is a linear mapping? Why?

Problem: Describe all polynomials $P$, with real numbers for their domain and range, for which $P(a + b) = P(a) + P(b)$.

Why is a constant term, $a_0$, not needed here?

Corresponding coefficients must be equivalent because the equivalency must hold for an infinite number of inputs.

Once you have a contradiction, one of your original premises—in this case, that there is a polynomial of the form $P(x) = r_1 x + r_2 x^2 + \cdots + r_n x^n$ which is an additive homomorphism, or there is more than one distinct zero—must be incorrect.

Problem: Show that if $F(a + b) = F(a) + F(b)$ and $r$ is a rational number, then $F(ra) = rF(a)$, you need to use previous results. You should know how to find values for unit fractions. You know how to find function values when the input is expressed as the product of an integer and an input whose function value you know.

6. This seems similar to problem 4, but mathematical induction would be difficult to use here. (Why?) The method you used to find function values for $x = \frac{m}{n}$ should help, though.

Problem: Show that if $F(a + b) = F(a) + F(b)$ and $r$ is a rational number, then $F(ra) = rF(a)$.

Problem: Can you determine a value for $F(\sqrt{2})$? For $F(\pi)$? Do problems 4 and 6 imply that $F(ka) = kF(a)$ for all real numbers $k$?

Problem: Which of problems 12a–12c could you still answer if you did not know that $T$ is a linear mapping? Why?

Problem: Describe all polynomials $P$, with real numbers for their domain and range, for which $P(a + b) = P(a) + P(b)$.

Why is a constant term, $a_0$, not needed here?

Corresponding coefficients must be equivalent because the equivalency must hold for an infinite number of inputs.

Once you have a contradiction, one of your original premises—in this case, that there is a polynomial of the form $P(x) = r_1 x + r_2 x^2 + \cdots + r_n x^n$ which is an additive homomorphism, or there is more than one distinct zero—must be incorrect.
zeroes. What would a graph with no zeroes look like? Next, assume there is only one zero, although that zero may appear multiple times. If there is only one zero, what must \( P(x) \) look like? Show that a polynomial of that form can be an additive homomorphism in only one case—that is, that it must be in the family you’ve already described.

- For a graphical argument, consider two values 1 unit apart, say, \( a \) and \( a + 1 \), where \( a \) is some rational number. How far apart are their function values? What if you had chosen two other values, 1 unit apart—how far apart are the function values for those? What does this tell you the graph of such a function must look like? What do you know about polynomial graphs that can help you conclude which families can be additive homomorphisms?

14. To prove statements like this, first assume the “if” part is true. Then use that assumption to show the rest must be true.

For part (a), you know that \( F \) is a linear mapping. Recall the definition of a linear mapping, and use it to find an equivalent expression for \( F(ax + by) \). (You should be able to show that \( aF(x) + bF(y) \) is an equivalent expression.)

For part (b), you know that \( F(ax + by) = aF(x) + bF(y) \). Can you then show that the conditions for being a linear mapping must be true for \( F \)? Perhaps some thoughtful numerical substitutions for \( a \) and \( b \) would help.

15. For part (a), use algebraic manipulation on \( x = \frac{n-m}{p-m} \) to get \( n \) on one side, alone.

For part (c), recall between what points the original formula interpolated. What values of \( x \) will give those points? What values will give points between them?

For part (d), you might approach the problem graphically, first. When you interpolated using linear combinations, what were you assuming the curve looked like (or almost like) between the two points? For what polynomials will the curve actually look like that? For a symbolic argument, you can refer to problem 13, but you have to justify why the answer here is not as restricted as the answer in problem 13. (This is also the answer to part (e).)

For example, the graph of \( y = (x + 1)^2 \) has only one zero, but it appears twice, once for each factor \( x + 1 \).

You don’t need to use \( P(x) = r_1x + r_2x + \cdots \) for this. Only use the fact that \( P(a + b) = P(a) + P(b) \).

Problem: (a) Show that if \( F \) is a linear mapping, then \( F(ax + by) = aF(x) + bF(y) \). (b) Show that if \( F(ax + by) = aF(x) + bF(y) \), then \( F \) is a linear mapping.

Problem: (a) Find an expression for \( n \) that uses \( x, m, \) and \( p \). (b) Write the formula in the form \( a(\text{linear combination of } p \text{ and } m) = \text{linear combination of } p \text{ and } m \). (c) What values should be used for \( x \)? (d) Describe all polynomials for which the interpolation is exact. Explain. (e) What in the linear interpolation formula is different from the definition of linear mappings and allows the polynomials for which the interpolation is exact to not be linear mappings?
4. Functions of two variables

As you worked through the first three sections of this chapter, you may have noticed times when more than one variable was used for a function or situation. In section 1, the blueness quotient (BQ) was based on both the number of blue beakers in a mixture and the total number of beakers in the mixture. In section 2, linear combinations often use two variables, such as in the linear combination $3x + 2y$. Some of the linear mappings in section 3 involved more than one variable, like the total cost of $x$ pounds of beans and $y$ pounds of rice.

In this session, you will explore functions of two variables (two input variables, one output variable). Some of the big questions are: How should linearity be defined for functions of two variables? Which features of linearity of one variable still apply in this new situation? What do those functions look like algebraically, graphically, and in tables?

The first problem context is adapted from the Mathematics in Context middle school curriculum.

1. Four pencils and 2 erasers cost 40 cents; 2 pencils and 5 erasers cost 60 cents.
   (a) Examine this chart, and explain what it will show when completed.

<table>
<thead>
<tr>
<th>number of pencils</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2 3 4 5 6</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2 40</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5 60</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

   (b) How can you use the chart (but no algebraic equations) to find the cost of 2 pencils and 1 eraser?
(c) Complete the chart, again without using algebraic equations. Record the steps you take.
(d) What is the cost of each eraser and each pencil? Could this be found from the table in more than one way?
(e) What are all the combinations shown in the chart which you could buy for exactly 55 cents? What are all the combinations you could buy for 30 cents? For any fixed amount of money, how could you describe all the combinations you could buy?

Reflect and Discuss

2. Compare your approaches with those of others. For the approaches that worked, explain what made them work.

3. Analyze the chart with traditional algebraic methods. Can the chart methods be interpreted algebraically?

You could think of the cost of pencils and erasers as a function of the numbers of pencils and erasers: \( C(p, e) \) is the cost of \( p \) pencils and \( e \) erasers.

4. (a) Before you filled out the chart, you were given the information \( C(4, 2) = 40 \) and \( C(2, 5) = 60 \). What is \( C(4 + 2, 2 + 5) \)?
(b) In general, if you knew \( C(p_1, e_1) \) and \( C(p_2, e_2) \), how could you find \( C(p_1 + p_2, e_1 + e_2) \)? Could this method have been used to fill the chart? Was it used by someone?
(c) Since you know \( C(4, 2) \), how could you find \( C(8, 4), C(12, 6), \ldots \)?
(d) In general, if you knew \( C(p, e) \), how could you find \( C(ap, ae) \) for a whole number \( a \)? Could this fact have been used to fill the chart?

In section 3, you learned that a linear mapping is a function \( F \) that satisfies the following conditions:

- \( F(a + b) = F(a) + F(b) \)
- \( F(ka) = kF(a) \)

As you saw, the quantities \( a, b, \) and \( k \) do not have to be real numbers. If the domain of \( F \) is a subset of \( \mathbb{R}^2 \) (ordered pairs of real numbers) and the range is a subset of \( \mathbb{R} \) (scalar, or one-
dimensional, real numbers), then the conditions for a linear map could be restated like this:

- \( F(x_1 + x_2, y_1 + y_2) = F(x_1, y_1) + F(x_2, y_2) \)
- \( F(kx, ky) = kF(x, y) \)

We could also define \( a \) and \( b \) to be two-dimensional vectors, and use vector addition and scalar multiplication. Abbreviate that is, \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\), and \( k(x, y) = (kx, ky) \).

Then the two conditions are

- \( F(v_1 + v_2) = F(v_1) + F(v_2) \)
- \( F(kv) = kF(v) \)

Note that this is exactly the definition of a linear mapping of one variable.

5. (a) Is \( C(p, e) \) a linear mapping?
   (b) In section 3, you saw that a linear mapping from \( \mathbb{R} \) into \( \mathbb{R} \) looks like \( F(x) = mx \) for some constant \( m \). What do you think a (continuous) linear mapping from \( \mathbb{R}^2 \) into \( \mathbb{R} \) looks like? Write a general description.

6. Is the function \( F(x, y) = 7x + y - 5 \) a linear mapping of two variables?

Reflect and Discuss

7. Should the function \( F(x, y) = 7x + y - 5 \) be considered a linear function? Why or why not?

The chickens and the eggs

You may have heard this joke problem before: “If a hen and a half can lay an egg and a half in a day and a half, how many eggs can 3 hens lay in 3 days?”

8. Most people’s first inclination is to say 3 eggs is the answer. This answer uses the idea of linearity, but incorrectly. How is linearity being used, and why is it incorrect in this case?

9. Try looking at the problem this way: Suppose you have an egg function \( E(H, D) \), which gives the number of eggs that \( H \) hens can lay in \( D \) days. Complete a table for the egg function, following the example of the pencils/erasers function. Use increments of half a hen on one axis, and increments of half a day on the other axis. What is the correct answer to the question?
10. Would you say from your table that the eggs function is linear? Why or why not?

11. In the following questions, $a$ and $b$ are positive. As you answer the questions, think about what they mean in terms of hens, eggs, and days. It might help to invent a barnyard scenario. Part a, for example, could be used to answer this question: If the number of hens on a farm increases by a factor of $a$, how does this affect the egg production?

   (a) Is $E(aH, D) = aE(H, D)$?
   (b) Is $E(H, bD) = bE(H, D)$?
   (c) Is $E(H_1 + H_2, D) = E(H_1, D) + E(H_2, D)$?
   (d) Is $E(H, D_1 + D_2) = E(H, D_1) + E(H, D_2)$?

12. Rephrase this question in terms of the egg function: How many eggs can 3 hens lay in 5 days? Use the identities you developed in problem 11 to answer it.

13. Write an explicit formula for the egg function.

14. Would you like to change your mind on whether the eggs function is a linear function? Why or why not?

15. Describe a family of functions $F$ from $\mathbb{R}^2$ to $\mathbb{R}$ that satisfy these conditions. Be sure to show that all functions in the family will satisfy them.

   - $F(ax, y) = aF(x, y)$
   - $F(x_1 + x_2, y) = F(x_1, y) + F(x_2, y)$
   - $F(x, by) = bF(x, y)$
   - $F(x, y_1 + y_2) = F(x, y_1) + F(x, y_2)$

   A function of two variables that satisfies these four identities is called a bilinear function.

Reflect and Discuss

16. In what way are the functions you described in problem 15 linear?

Graphs

The graph of a function of one variable uses two axes: one for the independent variable (input) and one for the dependent variable (output). In the same way, the graph of a function from $\mathbb{R}^2$ into $\mathbb{R}$ needs 3 axes: one for each independent variable, and one for the dependent variable. Often, the independent variables are called $x$ and $y$, and the dependent variable is called $z$. The $x$-$y$ plane is usually pictured horizontally, with the $x$-axis pointing towards you and the $y$-axis to the right; the $z$-axis is then pointing up. Here, the point $(-3, 4, 2)$ has been plotted.
The set of all possible input variables fills up the $x$-$y$ plane (or at least a region of it). The value of the function $z = F(x, y)$ is plotted at a height $z$ above the $x$-$y$ plane (or below it if $z$ is negative), directly over the point $(x, y)$.

17. (a) If you have graphing technology that will show three-dimensional graphs, graph the pencil-eraser cost function, $C(p, e)$.

If you don’t have that technology available, make a model of the graph using manipulatives: Cuisenaire rods, centimeter cubes, Unifix or Multifix cubes, or Legos. If you use Cuisenaire rods or centimeter cubes, a rod or stack of cubes should just fit on the square of the table corresponding to $(p, e)$. Choose your height scale carefully; if you use a centimeter to represent each penny of cost, the stack will be much too high. The top surface of the rods or stacks represents the graph of the cost function.

(b) How could you geometrically describe the graph? It helps to look at it sideways, in “profile.”

(c) Does the idea of the slope of a graph seem to apply to this graph? How?

(d) If the function were extended to include negative pencils and erasers, what would the graph look like?

Using your graph in problem 17, consider combinations for which the total cost is 55 cents, that is, solutions to the equation $C(p, e) = 55$.

18. What does the solution look like on the graph?
19. Compare this equation to an equation in one variable, for example, $5p = 55$.
   (a) How many solutions does each equation have?
   (b) When you graph the functions for each $(C(p, e)$ and say $D(p) = 5p)$, what dimension is the result? What dimension does the graph of the solution have?

20. What would the graph of $E(H, 0)$ look like? What would the graph of $E(H, \frac{1}{2})$ look like? What about the graph of $E(H, \frac{5}{2})$? Your table from problem 9 might help you decide.

21. What would the graph of $E(0, D)$ look like? What would the graph of $E(\frac{1}{2}, D)$ look like? What about the graph of $E(\frac{5}{2}, D)$?

22. What would the graph of the egg function $(E)$ from problem 9 look like? You might want to create a graph using manipulatives or software.
Ways to think about it

1b. Consider how these quantities compare to quantities you already know. Just thinking about the situation of buying pencils and erasers, how will the respective costs compare?

1c. Again, compare quantities with those you already know (including the one you found in problem 1b).

4b. Suppose you know the cost of a single pencil and a single eraser, as well as the cost for \( p_1 \) pencils and \( e_1 \) erasers. How can you find the cost when you buy \( p_2 \) pencils? What is \( e_2 \) in this case? Is there a particular \( p_2 \) to add (over and over) that will help you fill part of the chart quickly and easily?

5b. A function from \( \mathbb{R}^2 \) into \( \mathbb{R} \) might have a rule expressed using the two input variables. What would that expression look like? Was the cost function one? If so, how much can you change that function and still have a linear mapping? If not, is there something that can be changed that will result in a linear mapping?

8. To see how linearity is being used, consider how the input variables are being changed and how the original output is changed to get the incorrect answer. What aspect of linearity is used?

To understand why it’s incorrect, it might help to consider changing only one of the input variables. For example, suppose the original number of hens work twice as long as in the original statement. How does that change the number of eggs laid?

9. This problem might be harder than you expect at first. A good strategy is to record everything that you know, including the trivial situations of 0 days, or 0 hens. Using the original numbers \( E(1\frac{1}{2}, 1\frac{1}{2}) = 1\frac{1}{2} \), keep one of the numbers constant and vary the other.

It may help you to recall the interpolation problem from session 2: You have three increments between 0 and \( 1\frac{1}{2} \). How much will each increment need to be?

After filling in what you can from that, you have new information. For example, you might have the number that 1 hen can lay in \( 1\frac{1}{2} \) days. Can you use the same method to find how many eggs 1 hen can lay for other lengths of time?
13. With a full table, you have plenty of examples to try to find a pattern from, but that pattern may not be obvious.

Again, consider fixing one of the variables (H or D) and try to find a pattern just within that column or row. Then consider if the pattern holds among other columns or rows. If not, do new (similar) patterns exist? Are they similar to the first pattern? It helps if you use a systematic approach—for example, start with the first column, then try the next column, and so on.

If you’re still getting stuck, try this. At the bottom of each column, write how the numbers in the column change. (How can you change the entry in row n to get the entry in row n + 1?) Write a separate formula for each column: $E(H, 0)$, $E(H, \frac{1}{2})$, $E(H, 1)$, and so on. Next, how do the formulas change? Remember that for $E(H, \frac{1}{2})$, $D = \frac{1}{2}$ (and similarly so for each of the formulas). How do the formulas you wrote depend on the value of $D$?

16. If you used the method described for problem 13, you might have noticed some aspects of linearity in the patterns you found.

18. Remember that on the graph, the value of $z$ at the point $(p, e)$ is the function value, $C(p, e)$. One way to answer this question is to find the solution. Can you write an equation for $C(p, e) = 55$? What do you think the graph of that equation would look like (in three dimensions)?

If you’re using a graph created from manipulatives, find a way to visually identify the points for which the function value is 55.

If you’re using graphing software, you might graph the solution along with the function. This would require rewriting the equation in terms of one of the input variables, and using that variable as the output. For example, $p = \ldots$.

22. Try to use your answers to problems 20 and 21 to visualize the graph. How do the graphs change as you increase $H$? How do they change as you increase $D$? This type of visualization may be difficult for some people, so even if you think you know, you might want to check your answer by creating a graph using software or manipulatives.

Problem: Write an explicit formula for the egg function.

Problem: In what way are the functions you described in problem 15 linear?

Problem: What does the solution look like on the graph?

Problem: What would the graph of the egg function ($E$) from problem 9 look like?
5. From cups to vectors

In sections 1–4, you have looked a lot at what it means for a function to be linear. You’ve also looked at linear combinations of numbers. Much of what you’ve worked on showed connections among topics from elementary school to college-level courses.

You’ve probably seen variations of the following problem in many guises, including postage stamps and coins.

**PROBLEM**

1. **The Measuring Cups Problem**

   Kathryn has a cooking pot and two measuring cups. One cup holds 4 fluid ounces, the other holds 6 fluid ounces. Neither cup has marks that allow Kathryn to measure less than these amounts. Can she measure 2 fluid ounces using these cups? Can she measure 14 fluid ounces? 7 fluid ounces? For each amount she could measure, explain how.

   The connection between the Measuring Cups Problem and linear combinations is fairly simple: Any amount that can be measured using the cups is a linear combination of 4 and 6, $4x + 6y$, where $x$ and $y$ are both integers.

2. Use your answers to problem 1 to find solutions for the equations $4x + 6y = 2$ and $4x + 6y = 14$.

3. Provide a rationale to explain why it’s impossible to find integers $x$ and $y$ so that $4x + 6y = 7$.

4. What amounts can be measured using Kathryn’s measuring cups? Indicate how you might create those amounts, and explain why no other amounts can be created.

   **Greatest common divisors**

   You might be surprised to know that the Measuring Cup Problem also has connections to greatest common divisors.

5. What is the greatest common divisor (GCD) of 4 and 6? How does this GCD appear in your answer to problem 4?

6. For the following pairs of cups, find the GCD of the two amounts. Then describe all amounts that can be measured using the cups.
   - (a) 2 ounces and 5 ounces
   - (b) 3 ounces and 6 ounces
   - (c) 8 ounces and 12 ounces
   - (d) 3 ounces and 13 ounces

*Equations involving linear combinations with integers, such as $4x + 6y = 14$ where $x$ and $y$ must be integers, are called *Diophantine equations* for Diophantus of Alexandria. Fermat wrote his famous claim, known as Fermat’s Last Theorem, in the margin of a copy of one of Diophantus’ notebooks.*

*Then again, maybe this isn’t surprising.*
Reflect and Discuss

7. How do you know that you have found all amounts that can be measured using the pairs of cups in problem 6?

There are two theorems that can be seen from your work with the Measuring Cup Problem:

**Theorem 1** The GCD of two counting numbers \(a\) and \(b\) can be written as a linear combination of the numbers, \(ax + by\), where \(x\) and \(y\) are integers.

**Theorem 2** A positive linear combination \(ax + by\), where \(a\) and \(b\) are counting numbers and \(x\) and \(y\) are integers, is a multiple of the GCD of \(a\) and \(b\).

8. What do these two theorems say about a generalization of the Measuring Cups Problem: What measurements can be made using cups of \(a\) ounces and \(b\) ounces, if \(a\) and \(b\) are counting numbers?

9. Frank had only one measuring cup, which measured 3 fluid ounces. He borrowed one of Kathryn’s cups, claiming that he only needed that one (along with his 3-ounce cup) to get all integer measurements. Which cup should he borrow? How do you know?

10. What do you know about any two pairs of cups for which all integer measurements can be found?

On what basis?

In section 4, you looked at linearity in \(\mathbb{R}^2\). While linearity can show up in functions that use ordered pairs as inputs, linear combinations are important, too. A **basis** for \(\mathbb{R}^2\) is a special set of ordered pairs that spans the space—that is, a set with which all other ordered pairs in the space can be created using linear combinations. For example, \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\) span \(\mathbb{R}^2\). An ordered pair \((a, b)\) can be expressed as \(ae_1 + be_2\). A set of ordered triples, each of the form \((a, b, c)\), might span \(\mathbb{R}^3\)—the space of all ordered triples.

11. If the set spans the given space, find a linear combination of the pairs or triples that produces a general pair or triple, \((a, b)\) for \(\mathbb{R}^2\) or \((a, b, c)\) for \(\mathbb{R}^3\).

(a) \(2, 12\) and \((3, 0)\) for \(\mathbb{R}^2\)

(b) \((10, 15)\) and \((2, 3)\) for \(\mathbb{R}^2\)

(c) \((1, 1, 0)\) and \((1, 0, 1)\) for \(\mathbb{R}^3\)
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(d) \((1, 1, 0), (0, 2, 1),\) and \((1, 0, 1)\) for \(\mathbb{R}^3\)

(e) \((3, 5), (2, 12),\) and \((5, 17)\) for \(\mathbb{R}^2\)

A spanning set of pairs or triples might be called a basis for the corresponding space. To be a basis, the set must be linearly independent. A set is linearly independent if for every subset, a linear combination is 0—that is, \((0, 0)\) or \((0,0,0)\)—only when all the multipliers are 0. That is, if \(\{v_1, v_2, \ldots, v_n\}\) is linearly independent, then

\[
a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0
\]

is true only if

\[
a_1 = a_2 = \cdots = a_n = 0.
\]

12. Which of the sets in problem 11 are linearly independent?

13. If a set is not linearly independent, it’s said to be linearly dependent. (Was anyone surprised by that definition?)

(a) Show that if one ordered pair or triple is a multiple of another, the two are linearly dependent.

(b) Show that if an ordered pair or triple can be written as a linear combination of two others, the three are linearly dependent.

14. Suppose you have two pairs, \((a_1, b_1)\) and \((a_2, b_2)\). To determine if they are linearly independent, you have to consider linear combinations for which \(x(a_1, b_1) + y(a_2, b_2) = 0\).

(a) Use that equation to write two equations that you can graph on a coordinate plane. Both equations will involve \(x\) and \(y\).

(b) Where are the graphs of the two equations guaranteed to cross? Can you make a decision on the independence of the pairs based on that intersection?

(c) Suppose the two lines cross at another place as well. What does high school algebra tell you about the two lines and their equations in that situation?

(d) Show that if the two lines cross in more than one place, the pairs are linearly dependent—that is, there’s a nontrivial way (a way that doesn’t involve just multiplying everything by 0) to combine the pairs and get \((0, 0)\).

(e) If \((a_1, b_1)\) and \((a_2, b_2)\) are linearly dependent, what’s true about the two lines \(L_1\) passing through \((0,0)\) and \((a_1, b_1)\) and \(L_2\) passing through \((0,0)\) and \((a_2, b_2)\)? Are these the same lines as the ones you found in part (b)?
15. A set of ordered pairs or triples must be linearly independent and span a space to be a basis. Which of the sets in problem 11 are a basis for the stated space?

16. Some of the sets in problem 11 weren’t a basis for the given space but were linearly independent. For each of those, describe the space for which it is a basis.

The results with these ordered pairs and triples can be generalized to other vectors, which have the form \((a_1, a_2, a_3, \ldots, a_n)\). If each \(a_i\) is real, the space is then \(\mathbb{R}^n\). Each space is called a vector space. (There are conditions that define a general vector space, but \(\mathbb{R}^n\) always meets those conditions.)

**Reflect and Discuss**

17. In what way is problem 16 similar to the Measuring Cups Problem on page 106?
Ways to think about it

3. The parity of a number is the number’s evenness or oddness. For this particular problem, consider the parity of the numbers involved. You want to add two numbers to get an odd number . . . .

6. If you had only one cup, what amounts could you measure with just that cup? Can you find a way to create the GCD of the amounts for the two cups? Put your answers to these two questions together.

7. Basically, this question is asking you to show that amounts other than the ones you answered in problem 6 cannot be measured. This is a more general version of problem 3. In that problem, the parity of the numbers was involved with the reasoning. Something similar will work here. (Remember, parity is evenness or oddness, and a number is even if it’s divisible by . . . . What was the GCD in the original cups problem?)

8. Try to connect the theorems to problem 6. For one of those cup pairs, what would a and b represent? What does a linear combination of those numbers represent? How does the GCD fit in?

The difference between the two theorems might seem subtle, but it’s the same issue as in problem 7. One theorem implies what’s possible, the other tells you that those are the only things possible.

10. Theorem 2 says that only multiples of the GCD of two numbers can be created from linear combinations of those numbers. All integers are a multiple of what number? (Depending on your mathematics background, you might be able to attach a name to this: What does it mean for that number to be a GCD?)

11. Create a general linear combination for the pairs or triples. For example, for (2, 12) and (3, 0), the linear combination might be $x(2, 12) + y(3, 0) = (a, b)$. From this, you can write two (or in some cases, three) equations. Solve them for $x$ and $y$, in terms of $a$ and $b$. (If there are three pairs or triples, there will be three variables to solve for. For triples, there will be three constants, $a, b,$ and $c$.)

12. Use the definition of linear independence to write an equation using the set—a linear combination set equal to $(0, 0)$ or $(0, 0, 0)$. From that equation, you should be able to write a system of two or three equations (depending on how many
coordinates there are) in two or three variables (depending on how large the set is). How many solutions does the system have? If the set is independent, how many solutions would the system have?

16. To be a basis for a space, the set has to span the space. What space does the set span? That is, what does a general pair or triple in the space look like? (Look back at the definition of span if necessary.)

17. Suppose that instead of pairs and triples, the objects discussed in this section had only one coordinate, and only integers can be used for multipliers in linear combinations. What would spanning give in that case? See what parallels you can draw between the two situations in terms of linear independence as well.

Problem: For each set in problem 11 that isn’t a basis for the given space but is linearly independent, describe the space for which it is a basis.

Problem: In what way is problem 16 similar to the Measuring Cups Problem?