Chapter IV

Pythagoras and Cousins

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Introduction

The interplay between geometry and algebra is a theme that goes back at least to the Greek mathematicians of Euclid’s time. Even without standard algebraic notation, mathematicians have always been taken with the connections between calculations and visual images. In this chapter, we’ll look at some algebraic questions that are motivated by what is perhaps the most famous theorem from plane geometry, the Pythagorean theorem. Along the way, you will see how frequently it appears in the context of problem solving and become familiar with some of its mathematical cousins, which are found by tweaking the features of the Pythagorean theorem in the spirit of Chapter I, *What is Mathematical Investigation?*

A second theme weaving through the five sections of this chapter is one with which teachers are very familiar: how you can make up problems for your students that involve “nice” numbers. The search for Pythagorean triples is one example of such a “nice” number quest, but there are many others that appear in the field of Diophantine Geometry. For example:

- Can you find points $A$, $B$, and $C$ on the plane with integer coordinates so that $\triangle ABC$ has integer side lengths?
- Are there any scalene triangles with integer side lengths and a 60 degree angle?
- Which integers are areas of right triangles whose side lengths are rational numbers?

Some of these questions are quite simple to solve and some are amazingly difficult. We will address some of them in later sections, but will leave the last for you to ponder in your free time. (It remains an open problem in mathematics, although much progress has been made in the past two decades!)
1. What would Pythagoras do?

Take some time to think about, and solve, the following “nice” number quest. Also, be sure to reflect on the methods and strategies you use.

PROBLEM

1. **Rational Points on the Unit Circle**
   
   There are four integer-valued points on the unit circle; namely (1, 0), (0, 1), (0, −1), and (−1, 0). Are there any other rational points on the unit circle? If so, find at least six rational points in the first quadrant that lie on the unit circle.

Consider the following problems, all of which share (at least) one of the big ideas from problem 1. Most of these problems were motivated by activities in the *Math Connections*, *Mathematics in Context*, and *Connected Mathematics* curricula.

2. Find all points in the plane which are exactly 5 units away from the point (−1, 3).

3. Old MacDonald has a huge cornfield which is shaped like a rectangle, with sides running north-south and east-west. He knows that the area of the field is 12 square miles and and that the distance from the northeast corner to the southwest corner is 5 miles. What are the possible dimensions of the cornfield? Be sure to confirm that you’ve found all of the possibilities.

4. If the lengths of the sides of a triangle are 4.5, 12.3, and 13.1, is the triangle’s largest angle an acute, obtuse, or right angle?

5. Find the perimeter of \( \triangle ABC \), shown below, given that \( CD = 12 \), \( BD = 9 \), and angles \( ADC \) and \( ACB \) are right angles.
6. In the figure in the margin, the legs and hypotenuse of a right triangle are the diameters of semicircles. How does the sum of the areas of the smaller semicircles compare to the area of the larger semicircle?

7. If you know that \( \sin(\theta) = -\frac{5}{13} \), what can you say about the value \( \cos(\theta) \)?

Of course, the main thread that runs through each of the previous problems is the Pythagorean theorem, one of the most recognizable mathematical results. In the rest of this chapter, you will consider various mathematical cousins of this theorem, so it is important to start by carefully considering its features.

**PROBLEM**

8. **The Pythagorean Theorem**

Carefully state the Pythagorean Theorem. Be sure that your statement of the theorem will be clear to anyone, even if they have never heard of the theorem.

The Pythagorean Theorem is such a recognizable part of the curriculum that it is often taken for granted. Did you find that your initial statement of the theorem was incomplete? In mathematics, it’s important to be very clear about your underlying assumptions. After all, if you’re not completely clear on the hypotheses of the theorem, you might be tempted to use it in situations in which it doesn’t apply or to apply it incorrectly.

However, there’s an ulterior motive behind asking you to carefully think about what the Pythagorean Theorem says—and doesn’t say. In later sections, you’ll be asked to alter the hypotheses of the Theorem, and tweak it any other ways, as well, in order to make new discoveries or analyze familiar results. In the remainder of this section, though, you’ll apply the Theorem to derive well-known (yet, unfortunately, easily forgotten) formulas, then use the Pythagorean theorem to solve some problems concerning right triangles with a specified hypotenuse.

**Where’s Pythagoras?**

One of the points of doing all of the problems at the beginning of this section is to see that the Pythagorean Theorem comes up in many other mathematical contexts besides problems like If the two legs of a right triangle are 5 cm and 8 cm long, what is the length of the hypotenuse?
When you were a student you may have been required to memorize the formula for the distance between two points in the plane and the general form of the equation for the circle centered at the point \((a, b)\) having radius \(r\). The next few problems ask you to derive these formulas using only the Pythagorean Theorem. While there's nothing wrong with knowing these formulas, isn't it also useful to be able to derive them in case you—or your students—forget them? And if we make a few mathematical connections in the process, so much the better!

9. Determine the distance between points \((a, b)\) and \((c, d)\) in the plane.

10. Derive the general equation for all points \((x, y)\) lying on the circle of radius \(r\) centered at the point \((a, b)\).

The final activities of this chapter ask you to investigate properties of right triangles having a specified hypotenuse. Start with a specific segment \(AB\), and imagine all right triangles \(ABC\) having \(AB\) for a hypotenuse. If you plotted all possible points \(C\) so that \(\triangle ABC\) is a right triangle, what kind of graph would you get? Several points on the “C-graph” for one choice of \(AB\) are shown below (some with their corresponding triangles).

11. Using the partial C-graph above as an example, what do you think the whole graph will look like? Be as specific as possible with your description.
12. Let’s check your conjecture. First, imagine that the hypotenuse $\overline{AB}$ has length 2. In particular, imagine that $A = (-1, 0)$ and $B = (1, 0)$. Derive an equation for the $C$-graph of $\overline{AB}$.

13. Does your conclusion in the previous problem depend on the length of $\overline{AB}$ (the segment which is the hypotenuse for the $C$-graph)? Is the shape of the graph the same no matter what hypotenuse you start with? Explain how you can tell.

You’ve now shown that the $C$-graph for $\overline{AB}$ is the circle centered at the midpoint of $\overline{AB}$ with radius $\frac{1}{2}AB$. What happens if, rather than looking at all of the points $C$ so that $\triangle ABC$ is a right triangle, you look at $M$, the midpoint of $\overline{AC}$?

PROBLEM

14. How about the $M$-graph?
The $M$-graph for $\overline{AB}$ is the set of all points $M$ which are midpoints of $\overline{AC}$ where $\triangle ABC$ is a right triangle. What does the $M$-graph look like for a given $\overline{AB}$? Carefully explain your conclusions and reasoning.

Of course, when defining the $M$-graph, we could have chosen the midpoints of $\overline{AB}$, instead. What will be the same and what will be different about the $M$-graph defined in that way? Will the two different $M$-graphs coincide? Will they even intersect? Think about this before continuing to the next section.
Ways to think about it

For your convenience, the problems are restated in the margin.

1. You’re looking for points, \((x, y)\), so that \(x\) and \(y\) are rational numbers and \(x^2 + y^2 = 1\). There are at least two fruitful approaches to this problem. On the one hand, you can use trial and error—pick a rational value for \(x\), then solve for \(y\) and see if it’s rational. After a few trials, you’ll gain some insight into how to choose \(x\) (and therefore \(y\)) so that \((x, y)\) is rational. The other solution method involves investigating under what circumstances \((\frac{a}{b})^2 + (\frac{c}{d})^2 = 1\) when \(a, b, c, d\) are rational numbers. If you’re not assuming that the fractions are reduced, you can assume that \(b = d\). It will probably help to “clear denominators” first.

2. There are 4 points which are relatively easy to find—those that are 5 units to the left, 5 units to the right, 5 units above, and 5 units below \((-1, 3)\). In order to find all such points, suppose \((x, y)\) was one of them. At first, imagine that it is above and to the right of the point \((-1, 3)\). How can you express the fact that \((x, y)\) is exactly 5 units away from \((-1, 3)\)? If all such points were plotted, what would the shape of the resulting curve be?

3. Sketch the cornfield and draw in the segment which is said to be 5 miles long. What’s the relationship between the length and width of the field and this diagonal distance?

4. If the angle were a right angle, how would you be able to tell? What relationship would be satisfied by the three side lengths? If that relationship is not satisfied, how can you tell whether the angle is smaller (or larger) than 90 degrees?

5. First, solve for \(BC\). How do triangles \(\Delta ADC\), \(\Delta CDB\), and \(\Delta ACB\) compare?
6. How does the area of a circle depend on its diameter? What’s the relationship satisfied by the circles’ diameters?

![Diagram of a circle divided into two semicircles]

7. What’s the relationship between $\sin(\theta)$ and $\cos(\theta)$? Do you recall (or can you derive) the connection between $\sin(\theta)$, $\cos(\theta)$, and points on the unit circle?

8. Be sure not to leave out any hypotheses. What are the key features of the theorem? To what context does it apply? Any variable you introduce should be explained.

9. Draw two points in the plane and build a right triangle whose hypotenuse is the segment connecting the points. Be sure that your distance formula does not depend on the locations of $(a, b)$ or $(c, d)$ (or both).

10. The circle in question consists of all points which are exactly $r$ units from the point $(a, b)$. Which $(x, y)$ satisfy that condition?

11. How might you “connect the dots” using the points given? Does the shape look familiar? Do you recognize any symmetry? Once you have recognized a shape, get specific—that is, describe important defining features of the shape.

12. Let $(x, y)$ denote an arbitrary point $C$ so that $\triangle ABC$ is a right triangle. Compute $AC$ and $BC$ in terms of $x$ and $y$. What other algebraic relationship is satisfied by $AC$ and $BC$? Now, derive a relationship between $x$ and $y$. What does the graph of this curve look like?

13. This problem is the same as the previous one, except now the length of $\overline{AB}$ is not given. Try letting $A = (-r, 0)$ and $B = (r, 0)$ and proceeding as in problem 12. Is the shape of this $C$-graph the same as the previous one?
14. Recall that the $x$-coordinate of the midpoint of a segment is the average (arithmetic mean) of the $x$-coordinates of the endpoints of the segment. The $y$-coordinate of the midpoint is computed analogously. It might help to first consider a specific example, as in problem 12. Let $A = (-1,0)$ and $B = (1,0)$. It would be reasonable to let $C = (x,y)$, but remember that you’re trying to find out about $M$, the midpoint of $AC$. If you let $M = (x,y)$, you can determine the coordinates of $C$ using the relationship of the midpoint to the endpoints of the segment. Then proceed as before and determine a relationship between $x$ and $y$. Once you have an answer for the case $A = (-1,0)$ and $B = (1,0)$, use that information to predict what will happen in the general situation. Then, check to see what happens when $A = (-r,0)$ and $B = (r,0)$.
2. Puzzling out some proofs

In section 1, you worked on several variations on a theme of Pythagoras, then showed that a number of useful formulas are due entirely to the Pythagorean Theorem. At the end of the section, you worked through activities involving right triangles with a given hypotenuse and showed that the third vertex of the triangle (in addition to the endpoints of the hypotenuse) lies on the circle centered at the midpoint of the hypotenuse. In this section, you will spend most of your time “decoding” Proofs Without Words by providing the details of a visual justification of the Pythagorean Theorem. These proofs will just scratch the surface, however. The book, The Pythagorean Proposition, by E. S. Loomis (currently out of print, but originally published by Dr. Loomis in 1940, then reprinted by the NCTM in 1968) contains more than 360 different proofs of the theorem!

Truth be told, Proofs Without Words are not always proofs. A more accurate description would be that they are mnemonic devices for visualizing, recalling, or reconstructing a proof of a given theorem. Here’s one such “proof” of the Pythagorean Theorem.

1. Show how the figure below can be used to provide a visual proof of the Pythagorean Theorem by showing that the area of the shaded square (the square of the hypotenuse of the right triangle shown) is equal to the sum of the areas of the other two squares (the sum of the squares of the legs of the triangle).
Which of these proofs would be accessible to your students?

The idea here is that the similarly shaded regions have equal area. As constructed, the outer border of each figure is a square and the triangles are all congruent right triangles. You’ll want to show that the quadrilaterals are squares.

In each of the following problems, determine how the figures provide a Proof Without Words of the Pythagorean Theorem. Be sure to explain all details. You may assume that the triangles with side lengths $a$, $b$, and $c$ are right triangles, but you may not assume other angles are right angles (that must be proven).

2. 

![Diagram](image)

3. 

![Diagram](image)

The proof that accompanies this picture is usually attributed to James Garfield, the twentieth president of the United States. It appeared in the April 1, 1876 issue of the New England Journal of Education when Mr. Garfield was a member of the U.S. House of Representatives. However, it was likely known at least a millennium earlier to Arab and Indian mathematicians.

4. 

![Diagram](image)
PROBLEM

5. Euclid’s Proof of the Pythagorean Theorem

The following paragraph (and associated figure) provides the gist of the argument Euclid used in The Elements to prove the Pythagorean Theorem (Proposition 47 of Book 1). Fill in the details completely and carefully.

As before, \(\triangle ABC\) is a right triangle with right angle \(\angle ACB\), and quadrilaterals \(ABDE\), \(CAIH\), and \(BCGF\) are squares. Let \(J\) be the foot of the perpendicular from \(C\) to \(DE\), and \(K\) is the intersection of \(AB\) and \(CJ\). Since rectangle \(DJKB\) and square \(BCGF\) have equal area and rectangle \(JEAK\) and square \(CAIH\) have equal area, \((AC)^2+(BC)^2=(AB)^2\).

Proofs with words

Not all proofs of the Pythagorean Theorem are Proofs Without Words, even those that require a picture! Here’s one, which requires a little bit of work to “decode.”

6. Finish the proof of the Pythagorean Theorem suggested by the following figure.

\[x + y = c\]
The last proof of the Pythagorean Theorem that you’ll consider (in this section) is similar to Euclid’s proof, but is chosen because it leads to a surprising generalization of the theorem.

7. Using the now familiar figure of the right triangle with appended squares, prove that the areas of the similarly shaded regions are equal, thus showing that the sum of the areas of the “$a$-square” and “$b$-square” equals the area of the “$c$-square.” Note that $G$ is the intersection of $\overrightarrow{HJ}$ and $\overrightarrow{DE}$, $GL$ contains $C$ and $N$, and $O$ is chosen so that $\triangle KMO \cong \triangle ABC$.

You’ve just gone through several proofs of the Pythagorean Theorem. It might seem like a lot, but there are 367 proofs of the theorem in *The Pythagorean Proposition*, referred to on page 121. Once a theorem is proven, why would someone want to find another proof—not to mention 366 more?

**Reflect and Discuss**

8. Why might it be useful for you to work through a number of different proofs of the Pythagorean Theorem? Would it be useful for your students to work through several proofs, too?
Pythagoras’s first cousins

In mathematics, it is often very fruitful to play the “What if?” and “What if not?” games with the hypotheses and context of a known result. Let’s take the Pythagorean Theorem as an example (now there’s a shocker!). The Pythagorean Theorem was originally stated, by Euclid for instance, in terms of areas: The area of the square on the hypotenuse is the sum of the areas of the squares on the legs. In fact, mathematicians of ancient Greece did not separate numbers from geometry—products, like squares, had to denote areas. What if, instead of building squares off the sides of a right triangle, you built some other sort of figure? As you showed in the first section of this chapter, if you build semicircles off the sides, the area of the semicircle built on the hypotenuse is equal to the sum of the areas of the semicircles built on the two legs. This brings up an obvious question, with a less than obvious answer. Which types of figures work like squares (or semicircles) and which figures don’t?

9. Construct equilateral triangles on the sides of a right triangle, as in the figure below. Is the area of the triangle on the hypotenuse equal to the sum of the areas of the triangles on the legs?

10. In the figure below, rectangles of equal height have been built off the sides of a right triangle. Is the sum of the areas of the smaller rectangles equal to the area of the larger rectangle?
11. Construct rectangles on a right triangle so that the base of each rectangle is a side of the triangle and the height of each rectangle is half the base. Is the area of the rectangle on the hypotenuse equal to the sum of the areas of the rectangles on the legs?

12. Among the previous few attempts at generalizing the Pythagorean Theorem by replacing squares with other figures, only problem 10 didn’t “work.” What about the hypotheses of that problem was different from those in problems 9 and 11? Make a conjecture, which is as general as possible, specifying the types of figures that can be placed on the sides of a right triangle so that the area of the figure on the hypotenuse is guaranteed to be equal to the sum of the areas of the figures on the legs.

That’s all for now. What other Pythagorean cousins have you thought of? In section 3, you will meet some more by exploring the question,

*What if $\triangle ABC$ isn’t a right triangle?* How does the sum of the areas of squares built off the legs compare to the area of the square built off the hypotenuse?
Ways to think about it

1. Can you imagine moving the puzzle pieces (formed by the shaded region) to fill in the two smaller squares (built off the legs of the right triangle)? Some of the pieces of the shaded square already lie in one of the smaller squares (they are shaded more darkly in the figure below). *Don't move them.* Notice that an additional segment has been drawn into the figure below. This is not an accident. Do you see how it helps with the puzzle fitting?

If it helps to visualize the situation, make copies of the figures provided below, then cut out and move the pieces around until they fit. (For instance, cut up the shaded square in the figure on the right, then fill the figure on the left with the pieces.) In the end, you’ll want to describe which pieces fit where, so you’ll probably want to add some labels to the figure.

**Problem:** Explain how the figure provides a proof of the Pythagorean Theorem. You may assume that the triangle at the “top” of the shaded region is a right triangle and that the quadrilaterals built off the sides of the triangle are squares.
Problems 2-4: Determine how the figures provide a Proof Without Words of the Pythagorean Theorem. Be sure to explain all details. You may assume that the triangles with side lengths $a$, $b$, and $c$ are right triangles, but you may not assume other angles are right angles (that must be proven).

2. You are given that the four triangles in the figure on the left (with side lengths $a$, $b$, and $c$) are right triangles. How do you know the quadrilateral in the figure on the left and the similarly shaded quadrilaterals in the figure on the right are squares?

3. Compute the area of the trapezoid in two different ways. The area of a trapezoid with parallel sides of length $x$ and $y$ is $\frac{1}{2}(x + y)h$, where $h$ is the perpendicular distance between the parallel sides (as in the figure).

4. You’ll need to compute the area in two ways, but first, make sure the triangles fit together as drawn. You may assume that the four triangles are right triangles, but how do you know that the outer quadrilateral is a square? What’s the shape of the interior quadrilateral? What are its dimensions?

5. To show that the area of DJKB is the same as the area of BCGF: Show that $\triangle CBD \sim \triangle FBA$ and observe that (and explain why) the area of $\triangle CBD$ is half the area of rectangle DJKB and the area of $\triangle FBA$ is half the area of rectangle BCGF (what are the height and base of each of these triangles?).

To show that the area of JEAK is the same as the area of CAIH: Repeat the above strategy with $\triangle CAE$ and $\triangle LAB$. 

Euclid’s Proof of the Pythagorean Theorem

As before, $\triangle ABC$ is a right triangle with right angle $\angle ACB$, and quadrilaterals $ABDE$, $CAIH$, and $BCGF$ are squares. Let $J$ be the foot of the perpendicular from $C$ to $DE$, and $K$ is the intersection of $AB$ and $CJ$. Since rectangle DJKB and square BCGF have equal area and rectangle JEAK and square CAIH have equal area, $(AC)^2 + (BC)^2 = (AB)^2$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{euclid證明.png}
\caption{Euclid’s Proof of the Pythagorean Theorem}
\end{figure}
6. First, note (and prove) that the three right triangles in the figure are similar to one another. How does this fact allow us to conclude that $a^2 = x(x + y)$ and $b^2 = y(x + y)$? What does that tell us about $a^2 + b^2$?

![Diagram](image)

$x + y = c$

7. In this proof, you first need to compare the area of parallelogram $FGCB$ to the area of square $ECBD$ and the area of parallelogram $BCOM$ to the area of rectangle $BNLM$. It will help to first confirm that $\triangle GCE \cong \triangle ABC$ (what does this say about the two parallelograms?). Next, compare parallelograms, rectangles, and squares on the right side of the figure.

![Diagram](image)

8. Did you learn anything new by considering these proofs? Are any of them memorable (or at least “remember-able”)?

**Problem:** Finish the proof of the Pythagorean Theorem suggested by the figure.

**Problem:** Using the now familiar figure of the right triangle with appended squares, prove that the area of the similarly shaded regions are equal, thus showing that the sum of the areas of the “a-square” and “b-square” equals the area of the “c-square.” Note that $G$ is the intersection of $\overline{HJ}$ and $\overline{DE}$, $GL$ contains $C$ and $N$, and $O$ is chosen so that $\triangle KMO \cong \triangle ABC$.

**Problem:** Why might it be useful for you to work through a number of different proofs of the Pythagorean Theorem? Would it be useful for your students to work through several proofs, too?
9. To compute the areas of the triangles, you need to know their heights (we already know the side lengths). What’s the height of an equilateral triangle with side length $s$? If you drop an altitude from a vertex to the opposite side, where does it intersect the “base”? How do you know?

![Equilateral Triangle](image)

Remember that you may use the Pythagorean Theorem in your proof—we’ve proved it enough already!

10–11. Say the rectangles’ lengths (a.k.a. the sides of the triangles) are $a$, $b$, and $c$. What are the areas of the two smaller rectangles? What is the area of the larger rectangle? Do you arrive at the expected result?

12. In problem 10, the heights are constant, while in problem 11, the heights are proportional to the corresponding lengths. How, then, do the figures compare to one another? Try some other figures to test your hypothesis.
3. Pythagoras’s second cousins

In the previous sections, you worked through several applications of the Pythagorean Theorem, analyzed a number of different proofs of the Theorem, and began to investigate a few Pythagorean cousins—results which are related, but not identical, to the Pythagorean Theorem. In this section, you will meet some more members of the Pythagorean family as you continue to play “What if . . .?” and “What if not?”

1. Suppose ∆ABC is a triangle and that \(a = BC\), \(b = AC\), and \(c = AB\). Fill in the table below, describing whether \((a^2 + b^2) - c^2\) is positive, negative, or zero depending on the measure of \(\angle ACB\). Provide brief justifications for your choices.

<table>
<thead>
<tr>
<th>(\angle ACB)</th>
<th>((a^2 + b^2) - c^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>acute</td>
<td></td>
</tr>
<tr>
<td>right</td>
<td></td>
</tr>
<tr>
<td>obtuse</td>
<td></td>
</tr>
</tbody>
</table>

2. Let’s investigate the expression \((a^2 + b^2) - c^2\) a little further. If \(a\) and \(b\) stay constant and \(\angle ACB\) varies from 0 to 180 degrees, describe the behavior of \((a^2 + b^2) - c^2\) by addressing the following questions.

(a) Does it increase, decrease, or oscillate?
(b) Can you think of a trigonometric function with behavior similar to that of \((a^2 + b^2) - c^2\)?

As \(\angle ACB\) moves from 0 to 180 degrees, \(\cos(\angle ACB)\) acts a lot like \((a^2 + b^2) - c^2\). Namely, both functions decrease from positive to negative, equalling zero when \(\angle ACB = 90\). Does \((a^2 + b^2) - c^2\) equal \(\cos(\angle ACB)\)? The table below lists the side lengths of several triangles, all containing a 60° angle between the sides of length \(a\) and \(b\). Therefore, \(\cos(\angle ACB)\) is the same in each case. However, \((a^2 + b^2) - c^2\) is clearly not constant, so \((a^2 + b^2) - c^2 \neq \cos(\angle ACB)\).

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>((a^2 + b^2) - c^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>7</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>7</td>
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</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>13</td>
<td>105</td>
</tr>
</tbody>
</table>

So how do \(\cos(\angle ACB)\) and \((a^2 + b^2) - c^2\) compare?
3. The side lengths, \( \angle ACB \), and \( \cos(\angle ACB) \) for several triangles are given in the following table. Fill in the final column with the value of \((a^2 + b^2) - c^2\), then make a conjecture concerning the relationship between \((a^2 + b^2) - c^2\) and \(\cos(\angle ACB)\) which takes each of these examples into account.

<table>
<thead>
<tr>
<th>( \angle ACB )</th>
<th>( \cos(\angle ACB) )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>((a^2 + b^2) - c^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>60°</td>
<td>1/2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>1/2</td>
<td>3</td>
<td>8</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>1/2</td>
<td>5</td>
<td>8</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>1/2</td>
<td>7</td>
<td>15</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>(\sqrt{3}/2)</td>
<td>3</td>
<td>5</td>
<td>(\sqrt{34 - 15\sqrt{3}})</td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>(\sqrt{2}/2)</td>
<td>3</td>
<td>5</td>
<td>(\sqrt{34 - 15\sqrt{2}})</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>1/2</td>
<td>3</td>
<td>5</td>
<td>(\sqrt{19})</td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>(\sqrt{3}/2)</td>
<td>4</td>
<td>7</td>
<td>(\sqrt{65 - 28\sqrt{3}})</td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>(\sqrt{2}/2)</td>
<td>4</td>
<td>7</td>
<td>(\sqrt{65 - 28\sqrt{2}})</td>
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<tr>
<td>60°</td>
<td>1/2</td>
<td>4</td>
<td>7</td>
<td>(\sqrt{37})</td>
<td></td>
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</table>

**PROBLEM**

4. **The Law of Cosines**

Prove the conjecture you made in problem 3.

You’ve now succeeded in generalizing the Pythagorean Theorem to the case when \(\triangle ABC\) is *not* a right triangle, showing that the difference between the sum of the areas of the squares built on two sides of the triangle and the area of the square built on the third side is a function of the lengths of the first two sides and the angle opposite the third side.

One version of the Law of Cosines states that if \(BC = a\), \(AC = b\), and \(AB = c\), then \(a^2 + b^2 = c^2 + 2ab \cos(C)\), where \(C\) represents \(\angle ACB\). Is this the version that you recall?

The following theorem, attributed to Pappus of Alexandria (c. 300-350), illustrates a startling fact that on the one hand
Pythagoras and Cousins

provides another proof of the Pythagorean Theorem and on the other hand is a generalization of the Pythagorean Theorem. In fact, the proof of the Pythagorean Theorem discussed in problem 7 of section 2 is a special case of Pappus’ Theorem. You are given the chance to prove the theorem in the Further Exploration materials.

**Pappus’ Theorem:**
Suppose that \( \triangle ABC \) is a triangle and that quadrilaterals \( BCED \) and \( CAGF \) are parallelograms (not necessarily similar to one another). Let \( X \) be the intersection of lines \( \overrightarrow{DE} \) and \( \overrightarrow{FG} \). Construct parallelogram \( ABYZ \) so that \( BY \) and \( AZ \) are parallel and congruent to \( XC \). Then the area of \( ABYZ \) is equal to the sum of the areas of \( BCED \) and \( CAGF \).

**Nontriangular cousins of Pythagoras**

Even though the Pythagorean Theorem is about triangles—specifically, right triangles—it has applications to other objects, in both two and three dimensions. And, as you probably guessed, the theorem has several nontriangular cousins, as well.

5. Find a relationship between the sum of the squares of the diagonals and the sum of the squares of the four sides of a rectangle.

6. Is there a relationship between the sum of the squares of the diagonals of a parallelogram and the sum of the squares of the sides?

Of course, “the square of the diagonal” is shorthand for “the square of the length of the diagonal.”

Surprise!

So far, we’ve only delved into 2-dimensional Pythagorean cousins. Does the Pythagorean theorem have any 3-dimensional generalizations or applications? Let’s see!

7. A rectangular box is 12 inches long, 9 inches wide, and 8 inches deep. What’s the furthest distance apart two points on the box can be from one another?

8. What’s the relationship between the three dimensions of a rectangular box (length, width, and height) and the length of its “diagonal”?

9. What’s the distance between the points \((1, 2, 3)\) and \((0, 4, 5)\) in space?
10. Find an equation for the sphere of radius 4 centered at the point $(1, -1, 2)$.

In problem 6, you proved that the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the two diagonals. Alternatively, one could say that the sum of the squares of the two different edge lengths is the average of the squares of the diagonals. Problem 8 shows that the same can be said of the relationship between the edges of a rectangular box and its diagonals.

11. Is there a relationship between the sum of the squares of the edges and the average of the squares of the diagonals in a “parallelepiped” (a slanting box, in which opposite sides are congruent parallelograms)?

Were you surprised that the solutions to problems 7–10 were all integer-valued? Is it sometimes important to have answers that “come out nice”? In the next section, you’ll investigate Pythagorean triangles, right triangles with integer-valued side lengths, and in section 5, you’ll discover ways of guaranteeing “nice” solutions for more problems (and you’ll learn some new, interesting mathematics along the way).
Ways to think about it

1. You can imagine that $\overline{AC}$ and $\overline{BC}$ form a hinge, allowing $\angle C$ to vary while keeping $a$ and $b$ constant. What happens as $\angle C$ varies?

[Diagram showing a triangle with a hinge at vertex C]

If you prefer a more algebraic approach to the problem, consider the figures below. Can you compute $(a^2 + b^2) - c^2$ using the Pythagorean Theorem and the fact that the figures can be partitioned into right triangles?

<table>
<thead>
<tr>
<th>$\angle ACB$</th>
<th>$(a^2 + b^2) - c^2$</th>
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<tbody>
<tr>
<td>acute</td>
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<tr>
<td>right</td>
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<tr>
<td>obtuse</td>
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2. If you built or imagined a hinge to solve problem 1, then it will probably help to use it here, too. As $\angle ACB$ grows, what happens to $c$ (the length of the hypotenuse)? Since $a$ and $b$ remain constant, what happens to $(a^2 + b^2) - c^2$?

In part (b), it might help to graph the sine, cosine, and tangent functions for angles between 0 and 180°. Alternatively, think about the “unit circle” definition of sine and cosine:

If the ray from the origin to a point $(x, y)$ on the unit circle makes an angle of $\theta$ with the positive x-axis (when measured in the clockwise direction), then $x = \cos(\theta)$ and $y = \sin(\theta)$.

3. Once you’ve filled in the column, take a look at the results. Do you notice any patterns? What do the results have to do with $\cos(\angle ACB)$? It might help to make a new table which “lumps together” triangles with the same value for $\angle ACB$. Is it significant that, when $\angle ACB = 45^\circ$, $(a^2 + b^2) - c^2$ has a $\sqrt{2}$ in it, and $(a^2 + b^2) - c^2$ has a $\sqrt{3}$ in whenever $\angle ACB = 60^\circ$?

Problem: Fill in the final column of the table, then make a conjecture concerning the relationship between $(a^2 + b^2) - c^2$ and $\cos(\angle ACB)$ which takes each of these examples into account.
4. There are at least two ways to prove that \((a^2 + b^2) - c^2 = 2ab \cos(\angle C)\)—we’ll call them the \textit{algebraic} and \textit{geometric} strategies.

- **Algebraic strategy:** This is a refinement of the algebraic strategy outlined in problem 1. Drop a perpendicular from \(A\) to \(\overline{BC}\) (letting \(d\) be its length, as in the figure) and apply the Pythagorean Theorem to the two right triangles that are created.

\[
\begin{align*}
&\text{acute case} \\
&\triangle ACD \quad \triangle ABC \\
&\text{obtuse case} \\
&\triangle ABD \quad \triangle ABC
\end{align*}
\]

Use the right triangles in the figure, along with the Pythagorean Theorem, to find a relationship between \(a^2, b^2, c^2\), and \(\cos(\angle ACB)\). In the acute case, \(a = e + f\), \(c^2 = d^2 + e^2\), and \(b^2 = d^2 + f^2\). Does this show us anything about \((a^2 + b^2) - c^2\)? The identity \(\cos(\theta) = -\cos(180 - \theta)\) might prove useful. What’s the relationship between \(f\) and \(\cos(\angle ACB)\)?

- **Geometric strategy:** Use the figure below as a guide to determine the relationship between the sum of the areas of the squares built off the shorter sides and the area of the square built off the long side of \(\triangle ABC\) when \(\angle ACB\) is acute.

You may assume that quadrilaterals \(BCHJ\), \(ABLN\), and \(CADF\) are squares, \(\overline{BG} \perp \overline{AC}\), \(\overline{CO} \perp \overline{AB}\), and \(\overline{AK} \perp \overline{BC}\).

You’ll need to express \(AB^2 - (AC^2 + BC^2)\) as a function of \(AB\) and \(\cos(\angle ACB)\). Can you show that the areas of similarly shaded rectangles are congruent? (First, compare \(\triangle ADB\) to \(\triangle ACN\) and \(\triangle BCL\) to \(\triangle ABJ\).) What are the areas of \(GEFC\) and \(KCHI\) in terms of \(\cos(\angle ACB)\)? (Note that \(\cos(\angle ACB) = \frac{CG}{BC} = \frac{CK}{AC}\).)
The figure for the obtuse case is given below:

As before, the quadrilaterals built off the sides of $\triangle ABC$ are squares, $BF \perp DE$, $AH \perp IJ$, and $CM \perp LN$. Can you show the areas of $ADFG$ and $AOMN$ are equal? (Compare triangle $ADB$ to $ACN$.) What about $OBLM$ and $HJBK$? (Compare triangle $BCL$ to $ABJ$.) Finally, determine the areas of $EFGC$ and $KHIC$, in terms of $\cos(\angle ACB)$?

5. Look for the hidden right triangles in the rectangle. Recall that in section 1, you showed that the two diagonals were equal (alternatively, take a second to remind yourself why it’s true!).

6. Try a few examples. If you have access to Dynamic Geometry software (Geometer’s Sketchpad, Cabri Geometry, a TI-92 calculator, or Cabri Jr. on a TI-83, for example), use it to generate some examples, but they’re also easy enough to sketch using (square) grid paper. Be sure that your proof takes all possible cases into account. Another way to think about the problem is to use the Law of Cosines (surprise!), since a diagonal partitions a parallelogram into two triangles. If you’re familiar with vectors and their properties (especially lengths and dot products), you can use vector methods instead to solve the problem.

Problem: Find a relationship between the sum of the squares of the diagonals and the sum of the squares of the four sides of a rectangle.

Problem: Is there a relationship between the sum of the squares of the diagonals of a parallelogram and the sum of the squares of the sides?
7. First, find the length of the diagonal of the base of the box. Use that information to find the diagonal we’re really interested in. In the process, be sure to convince yourself (and anyone else) that this diagonal is what you’re looking for.

![Diagram of a rectangular box with a diagonal marked]

8. This is just the abstract version of the previous problem. Call the dimensions $a$, $b$, and $c$, and let $d$ denote the diagonal. Can you solve for $d$?

9. Imagine a box with the two given points at the ends of the “diagonal.” What would the coordinates of the corners be? Alternatively, what are the dimensions of the box?

10. All of the points on the sphere will be exactly 4 units from the center. It might be helpful to derive and discuss the 3-dimensional distance formula (of which the work in problem 9 is a special case).

11. As suggested in the paragraph preceding the problem statement, this investigation is the 3-D analogue to problem 6. The methods are analogous as well. Try some examples (your facilitator may have some for you to investigate).
4. Pythagorean triples (and cousins)

In this section, the Pythagorean cousin you’ll investigate led to what is probably the most famous mathematical problem of the past four centuries—perhaps of all time. In fact, this problem was the genesis of an entire branch of mathematics, Algebraic Number Theory. But we’re getting ahead of ourselves. First things first.

If you ask someone to give you an example of the side lengths of any right triangle, they’ll probably think you’re kidding. If you’re able to convince them that you’re serious, then the most common response you’ll get will probably be “3, 4, and 5.” The ordered triple $(3, 4, 5)$ is an example of a Pythagorean triple, which can be defined in a couple of equivalent ways:

- **Algebraically**, we say that $(a, b, c)$ is a Pythagorean triple if $a$, $b$, and $c$ are integers and $a^2 + b^2 = c^2$.
- **Geometrically**, $(a, b, c)$ is a Pythagorean triple if $a$, $b$, and $c$ are integers and also the lengths of the legs and hypotenuse of a right triangle.

1. You probably know several more Pythagorean triples besides $(3, 4, 5)$. List as many as you can think of.

Did you have trouble coming up with other triples? Don’t worry, by the end of the section, you’ll have a method to generate as many triples as you want! But first, let’s consider some properties of Pythagorean triples.

**PROBLEM**

2. **Can Pythagorean Triples Ever be Odd?**
   How many even entries can any Pythagorean triple have? (Are 0, 1, 2, and 3 all possible?) Can the hypotenuse ever be the only even side length in a Pythagorean triangle? Prove that you’re right.

   It can be useful to know some Pythagorean triples, especially when solving problems involving right triangles, since the sides often have integer lengths. Students can solve the problem below without knowing ahead of time that the side lengths are all integers, but it’s even easier to solve if you know the
Pythagorean triples that are hidden within the problem. Here’s a type of problem that occurs in many curricula.

3. Determine the perimeter of $\triangle ABC$ (in the margin), given that $AC = 13$, $CD = 12$, and $BD = 16$. Repeat the problem assuming that $AC = 15$ (while $CD$ is still 12 and $BD$ is still 16).

While we’re often happy for students to recognize Pythagorean triple patterns, there might also be occasions in which we want them to do the required arithmetic to solve for one of the values in the equation $a^2 + b^2 = c^2$. How, then, can you find more Pythagorean triples in order to construct problems like the previous one, but with side lengths that aren’t part of familiar Pythagorean triples? One way would be to use triples which are multiples of the original ones by multiplying each of the given sides by a number, like 7, then ask students to determine the perimeter. It’s unlikely that a student would automatically recognize that 84 is the missing entry in the triple (35, ?, 91), so they’d have to solve the problem algebraically. You could also use trial-and-error: Pick two integers for $a$ and $b$ and see if $\sqrt{a^2 + b^2}$ is an integer. Alternatively, you could pick integer values for $a$ and $c$ and check whether $\sqrt{c^2 - a^2}$ is an integer.

Are there any ways to narrow our search? Are there properties Pythagorean triples must satisfy that we can apply? Let’s investigate.

The table below lists several related Pythagorean triples. Do you see how they are related? Can you find the next few triples in the table?

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<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
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<td>13</td>
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Check that the triples you add to the table are Pythagorean triples.

4. Find a pattern in the above table, then determine the next 3 triples by following the observed pattern.

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</table>
5. Use the pattern found in problem 4 to create a formula that will generate infinitely many Pythagorean triples. Prove that your formula always gives a Pythagorean triple.

6. Here’s another table of Pythagorean triples, which follows a different pattern. Find the pattern, then fill in the next three triples in the table by following the pattern.

<table>
<thead>
<tr>
<th>a</th>
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<th>c</th>
<th>a</th>
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7. Based on the previous table, guess another formula that will generate infinitely many Pythagorean triples. Prove that your formula always gives a Pythagorean triple.

As mentioned earlier, one of the reasons that teachers might want to know a lot of Pythagorean triples, or at least know how to find a lot of them, is that they could then create problems involving Pythagorean triangles for their students.

8. Find two different Pythagorean triangles with legs of length 15, to rewrite the hypotheses of the first half of problem 3 so that $CD = 15$, instead of 12. That is, find integer lengths for the sides marked with a “?”.

Problems 4–7 provided some ways of finding infinitely many Pythagorean triples—some, but not all of them. There is a way of finding every single Pythagorean triple, though. In Further Exploration, you’ll have the opportunity to work through the derivation of the following formula, which generates all Pythagorean triples.

A Pythagorean triple formula

$$(n^2 - d^2, 2nd, n^2 + d^2)$$

is a Pythagorean triple whenever $n$ and $d$ are positive integers and $n > d$. Not only is every triple of this form a Pythagorean triple; every Pythagorean triple is similar to a triple of this form.

Before continuing, take a moment to ponder this amazing fact. The Pythagorean triple formula provides a way to find every Pythagorean triple there is! The next problem asks you to confirm that the formula always gives Pythagorean triples.
9. Show that if \( n, d, \) and \( k \) are positive integers and \( n > d \), then \((k(n^2 - d^2), 2knd, k(n^2 + d^2))\) is a Pythagorean triple.

10. Use the formula to find five Pythagorean triples that don’t appear in the tables from problems 4–7.

11. Use the **Pythagorean triple formula** to find all Pythagorean triangles with at least one side of length 12.

**Properties of Pythagorean triples**

Here’s a list of Pythagorean triples you may have seen in the *Ways to think about it* section for problem 2.

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That is, for what integers, \( d \), do you think it’s true that every Pythagorean triple has an entry divisible by \( d \)?

12. You’ve shown that every Pythagorean triple has an entry divisible by 2. What other properties involving divisibility seem to be satisfied by all of the Pythagorean triples in the table? You don’t have to prove your conjectures, yet. You are given the opportunity to do that in the *Further Exploration* materials.

If \((a, b, c)\) is a Pythagorean triple and \(a, b, \) and \(c\) share no common factors greater than 1, then \((a, b, c)\) is called a **primitive** Pythagorean triple.

13. Suppose \((a, b, c)\) is a primitive Pythagorean triple and \(b\) is the even entry. What can you say about the difference between \(b\) and \(c\)?

14. Is it true that if \((a, b, c)\) is a primitive Pythagorean triple, then at least one of \(a, b, \) or \(c\) is a prime?

Using your knowledge of Pythagorean triples to construct problems for students with numbers that come out nice (as in problems 3 and 8) is one of Pythagoras’ many “classroom cousins.” You’ll investigate some more of these in the final section of this chapter.
Ways to think about it

1. There are infinitely many possibilities here—be sure to check that your triples are Pythagorean triples. If you have trouble coming up with triples that work, can you think of any that are similar to (3, 4, 5)? If all else fails, try out some right triangles with integer-valued legs and check to see if the hypotenuse has integer length.

2. From the examples you generated for problem 1 and in the table below, there appear to be two choices: either all three entries are even or exactly one of them is even. Use the fact that every odd integer can be represented as $2k + 1$ for some integer $k$ and every even integer can be represented as $2n$ for some integer $n$.

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Some Pythagorean Triples

Looking at the entries in the table provided, or creating some more Pythagorean triples with the help of the formula, leads to the conjecture that $c$ can’t ever be the only even entry in a triple. Why is that? Suppose $(a, b, c)$ is a Pythagorean triple and that $a$ and $b$ are both odd. Then, since at least one of $a$, $b$, and $c$ must be even, $c$ must be the even one. Given that $c$ is even, by what number (larger than 2) can you guarantee $c^2$ will be divisible? If $a$ and $b$ are both odd, is it possible for $a^2 + b^2$ to be divisible by this number?

3. Notice that the figure can be partitioned into a number of right triangles and that you know some of the side lengths of these triangles already. Use the Pythagorean theorem, or your memory of Pythagorean triples, to find the lengths of the other sides.

Problem: You probably know several more Pythagorean triples besides (3, 4, 5). List as many as you can think of.

Can Pythagorean Triples Ever be Odd?
How many even entries can a Pythagorean triple have? Can the hypotenuse ever be the only even side length in a Pythagorean triangle? Prove that you’re right.

Problem: Determine the perimeter of triangle $ABC$, given that $AC = 13$, $CD = 12$, and $BD = 16$ (as in the figure). Repeat the problem if $AC = 15$ (while $CD$ is still 12 and $BD$ is still 16).
4. You’ve probably noticed that $a$ is odd in each entry, so $a^2$ is odd, as well. Do you notice how $b$ and $c$ compare to one another in each triple listed? What do you know about the difference of two consecutive squares? In trying to find a triple corresponding to $a = 15$, think about how $15^2$ can be the difference of two other squares.

5. The conjecture you make will depend on the pattern(s) you found in the table and the method you chose to fill in the next three entries in the table for problem 4. Two useful observations about the table are that $a$ is odd and that $b$ and $c$ are consecutive. If $a$ is odd, you can write $a = 2n + 1$. Letting $n$ range over all positive integers will provide infinitely many triples. Once you know that $a = 2n + 1$, what must $b$ and $c$ equal in order for $(a, b, c)$ to be a Pythagorean triple? If $b$ and $c$ are consecutive, then $c = b + 1$, since $c$ must be larger than $b$. What about the relationship between $b$ and $c$ guarantees that they can’t share a common divisor greater than 1? Can you express the entries of row $k$ in the table in terms of $k$?

6. Notice the pattern in the $a$ terms of the table, as well as the relationship satisfied by the $b$ and $c$ terms. In addition to being equal, what is true about $b + 1$ and $c − 1$ in each case? As mentioned in the margin, it’s important to check that the triples you find really are Pythagorean triples.

7. The key to the formula, as in problem 5, is representing the key features of the examples given in the table. If $b + 1$ is a square, then $b = n^2 − 1$ for some integer, $n$. If $c = b + 2$, then $c = n^2 + 1$. What does this say about $a$? Alternatively, what happens if you start with the assumption that $a$ is a multiple of 4? What does that say about $b$ and $c$? Can you express the entries of row $k$ in the table in terms of $k$?

8. Find two different triples with either $a$ or $b$ equalling 15, then label the two triangles in the figure with the corresponding values of the other lengths. Use the tables from the session or other Pythagorean triples you know. Once you’ve done that, restate problem 3 with the appropriate values for $AC$ and $BD$. 

![Diagram of a triangle with labels A, D, B, C, and question marks.]
9. What does it mean to say that \((a, b, c)\) is a Pythagorean triple? Is the definition satisfied by every triple of the form \((k(n^2 - d^2), 2knd, (k(n^2 + d^2))\)? You can use specific choices for \(k, n, \) and \(d\) initially, but you need to show that every triple of the stated form is a Pythagorean triple.

10. This is just a matter of carefully using the formula. While it’s possible to just choose \(k, n, \) and \(d\) at random, you can save some time by choosing the values systematically. For instance, start with \(k = 1\), then let \(n\) and \(d\) vary in some specific pattern which is easy to keep track of. Remember that the first two terms in a Pythagorean triple are interchangeable (since they give rise to congruent Pythagorean triangles).

11. Use the Pythagorean triple formula. For what values of \(k, n, \) and \(d\) could \(2knd = 12\)? Could \(k(n^2 - d^2)\) ever equal 12? What about \(k(n^2 + d^2)\)? It might help to first pick a \(k\) and try to find \(n\) and \(d\), then pick another \(k\), and so on. What are the possible values of \(k\) that make sense to try? If you are systematic, you can be sure that you’ve looked at all the possibilities, then can say with confidence that you’ve found all such triples.

12. There are a number of possible conjectures. Look at each triple one at a time. What can you say about the entries? Must there always be an odd term? An even term? A multiple of 4? A multiple of 3? What else do you see?

13. You can get some ideas from the tables. Once you have a conjecture, use the Pythagorean triple formula to describe all primitive triples (what must \(k\) be in this case?) and compute the difference in question. You can create more specific triples to check your conjecture or use the algebraic formula to characterize all possible differences.

14. First, check to see what the tables tell you. Be sure the triples you check are primitive. Use the formula to generate some more triples. How might you prove the conjecture if you can’t find any counterexamples? Do you hope it’s not true?

**Problem:** Show that if \(n, d, \) and \(k\) are positive integers and \(n > d, \) then \((k(n^2 - d^2), 2knd, k(n^2 + d^2))\) is a Pythagorean triple.

**Problem:** Use the formula to find five Pythagorean triples that don’t appear in the table on page 143.

**Problem:** Find all Pythagorean triangles with at least one side of length 12.

**Problem:** What other properties involving divisibility seem to be satisfied by all of the Pythagorean triples in the table?

**Problem:** Suppose \((a, b, c)\) is a primitive Pythagorean triple and \(b\) is the even entry. What can you say about the difference between \(b\) and \(c\)?

**Problem:** Is it true that if \((a, b, c)\) is a primitive Pythagorean triple, then at least one of \(a, b, \) or \(c\) is a prime?
5. More classroom cousins

In this section, you’ll apply some of what you learned in previous sections to creating useful classroom activities for you and your students. But first, in the spirit of the What is Mathematical Investigation? chapter, let’s consider what happens when we alter one of the features of another cousin. When looking for Pythagorean triples, you ask which integer squares are the sum of two integer squares. What about the difference of two squares? Although it’s not obvious, this turns out to be related to a Pythagorean classroom cousin.

**PROBLEM**

1. **The Difference of Two Squares**
   Which counting numbers can be expressed as the difference of the squares of two counting numbers?

And now, some more classroom cousins.

If you have taught geometry or trigonometry, the next problems are probably similar to some you’ve given in class. Take a few minutes to solve them in order to be aware of what’s involved when students work on them. Later, we’ll discuss the creation of these problems.

2. Determine the perimeter of right triangle $ABC$ (see the figure in the margin), given that $CD = 12$, and $AB = 25$.

3. Determine the perimeter of the triangle having vertices at the points $(1,2)$, $(10,14)$, and $(5,2)$.

4. Compute the area of the triangle having side lengths 3, 7, and 8, given that one of its angles measures 60 degrees. Repeat the problem with the assumption that the side lengths were 7, 13, and 15 (and one of the angles is still 60 degrees).

5. A rectangular box is 12 inches long, 9 inches wide, and 8 inches deep. How far apart are the lower, left, front corner and the upper, right, back corner?

Did you notice that all of the numbers in these problems were integers? That’s not an accident. While there are certainly cases where “messy” numbers are appropriate, there are times when it’s preferable for the numbers in the question and solution to “come out nice.” In the article “Meta-Problems in
Mathematics,” Al Cuoco wrote,

*I have a conjecture: A great deal of classical mathematics was invented by teachers who wanted to make up problems that come out nice. Problems that come out nice allow students to concentrate on important ideas rather than messy calculations. They give students feedback that they are on the right track. They are easier to correct.*

p. 373

But how do you construct problems that come out nice but aren’t the ones the students have already seen? In section 4, you learned how to find all right triangles with integer sides and applied this knowledge to create triangles with all three sides and at least one altitude having integer length. In this section, you will apply your knowledge of Pythagorean triples to find other classroom cousins. Specifically, you will now address classroom cousins (or meta-problems), corresponding to the creation of problems 2–5 and the big questions below.

- How can you find right triangles so that all three sides and the altitude to the hypotenuse have integer lengths?
- How can you find triangles with integer coordinates and integer sides?
- How can you find triangles with integer sides and a 60° angle?
- How can we find rectangular boxes with integer-valued side lengths and diagonal?

**Sides, altitudes, and vertices**

- How can you create right triangles so that the sides and the altitude to the hypotenuse are integer valued?
- How can you find triangles with integer coordinates and integer sides?

Look back to see how the triangles in problems 2 and 4 were constructed. In problem 2, what’s the relationship between the two subtriangles that are formed by the altitude? How does that guarantee that $\angle ACB$ will be a right angle? In problem 4, what’s the length of the altitude to the edge with endpoints (1, 2) and (5, 2)? What’s special about the lengths of the pieces into which the altitude splits the segment?

6. Rewrite problem 2 so that $CD = 60$ and the resulting triangle has integer sides (you’ll still need to specify $AB$).
PROBLEM

7. **Pythagorean Triangles with Integer Altitudes**
   Describe a method to create infinitely many Pythagorean triangles having altitudes with integer length.

8. Find 3 noncongruent triangles that have integer coordinates and sides. They don’t have to be right triangles.

60° triples and triangles

*How can you find triangles with integer sides and at least one 60° angle?*

The following formula generates the so-called 60° triples (triangles with integer sides and at least one 60° angle). You can derive the formula in the *Further Exploration* materials.

<table>
<thead>
<tr>
<th>A 60° triple formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2nd - d^2, n^2 - d^2, n^2 - nd + d^2)) is a 60° triple whenever (n) and (d) are positive integers and (n &gt; d). Not only is every triple of this form a 60° triple; every 60° triple is similar to a triple of this form.</td>
</tr>
</tbody>
</table>

9. Show that all triangles with side lengths given by the above formula do, in fact, contain a 60° angle. Between which two sides is the angle?

10. Find 3 noncongruent triangles that have integer sides and exactly one 60° angle.

Three-dimensional cousins

We finish this section with one final meta-problem:

*How can you find rectangular boxes with integer-valued side lengths and diagonal?*

You’ve already shown that if a rectangular box has length \(l\), width \(w\), and height \(h\), then the length of its diagonal is \(\sqrt{l^2 + w^2 + h^2}\). This is one of the many Pythagorean cousins we’ve met in this module. Thus, finding boxes with integer-valued sides and diagonal amounts to finding integer solutions to the equation \(d^2 = l^2 + w^2 + h^2\). Go back over your solution to problem 5, which is repeated in the margin. Does this problem give you any ideas about how to generate more such examples?
Problem 5 worked out “nicely” because the diagonal of the base of the box had an integer length that was simultaneously the hypotenuse of one Pythagorean triangle and a leg (along with the height of the box) of another Pythagorean triangle. This property is significant, since it gives you something to work with when you try to construct more such boxes.

11. Find 3 other boxes with dimensions satisfying the property that the three dimensions and diagonal are integer-valued.

When you find counting numbers $l$, $w$, $h$, and $d$ that satisfy $l^2 + w^2 + h^2 = d^2$, as you did in the previous problem, you’re also finding counting numbers that are solutions to the equation $l^2 + w^2 = d^2 - h^2$, which leads to the final activity of this session—in fact, it’s the last activity of the module.

PROBLEM

12. Sums and Differences of Squares
Which numbers can be expressed as the sum of the squares of two counting numbers and also as the difference of the squares of two counting numbers? Describe a method for generating infinitely many such numbers.

The Difference of Two Squares

Which counting numbers can be expressed as the difference of the squares of two counting numbers?

Problem: Determine the perimeter of right triangle $ABC$, given that $CD = 12$, and $AB = 25$.

Problem: Determine the perimeter of the triangle having vertices at the points $(1, 2)$, $(10, 14)$, and $(5, 2)$.

Problem: Compute the area of the triangle having side lengths 3, 7, and 8, given that one of its angles measures 60 degrees. Repeat the problem with the assumption that the side lengths are 7, 13, and 15 (and one of the angles is still 60 degrees).

Ways to think about it

1. Start by making a list of the first 10 or so nonzero squares, then look at the differences of these squares. Alternatively, make a list of the counting numbers up to 20 and see which of these can be expressed as the difference of nonzero squares. Do you see any patterns in the list of counting numbers which are, or aren’t, the difference of nonzero squares? Which even numbers are differences of squares? Which odd numbers? Enlarge your list, if necessary, to see the pattern.

Once you have a conjecture, look carefully at how the list is structured—again, look at evens and odds separately. Pick a number just beyond your list and see if you can guess, from the preceding pattern, how to express it as a difference of two squares.

2. Label the sides as shown below. What are the relationships between $x+y$, $w$, and $z$; between $w$, $CD$, and $x$; and between $y$, $CD$, and $z$ (remember that $\triangle ABC$ is a special kind of triangle)? How are the three triangles related to one another? In particular, how are $\frac{w}{x}$ and $\frac{x+y}{w}$ related? What about $\frac{z}{y}$ and $\frac{x+w}{z}$? Solve for $x + y + w + z$ after determining the value of each variable first.

It’s also possible to solve for $w$ and $z$ by computing the area of $\triangle ABC$ in two different ways (one in terms of $w$ and $z$).

3. Of course, you need to determine the lengths of the segments created by these three points.

4. Which angle measures 60°? Be careful, you can’t assume the figure is drawn to scale. Which one of the angles (small, medium, large) must it be? Why? Then, drop a perpendicular from the vertex at the 60° angle and you’ll have a nice 30-60-90 triangle to work with in order to determine the length of that altitude.
5. It will probably help to make a sketch of the box. Since we’re claiming this is a Pythagorean cousin, there should be some right triangles you can work with. Where are they?

6. This problem is similar to problem 8 of the section 4 Activities and Explorations, but the resulting triangle is a right triangle here. What must the relationship be of the two subtriangles if $\angle ABC$ is to be a right angle? How are the two subtriangles in problem 2 related to one another? Can you find two similarly related triangles that share a leg of length 60?

7. The general method should mirror your solution to the previous problem. If necessary, go back over the strategy you used and see how it can be generalized.

8. Look for similarities between the features of triangles from problems 2 and 3, and problems 3 and 8 in section 4. How can your work on these problems help you on this one?

9. What does the Law of Cosines say about triangles whose lengths come from the formula? Which angle (that is, between which sides) is the 60° angle? This is a difficult question, since $n$ and $d$ are not given. Compute, then factor, the three differences between side lengths. Why does the location of the angle depend on whether or not $n$ is greater than, less than, or equal to $2d$? Be sure to be careful with your arithmetic.

10. Use the formula to find some 60° triples.

11. This problem is related to problem 6, except here you need to match the hypotenuse of one triangle with a leg of another. One way to solve this problem is to look at the lists of triples you already have and hope for some “matches.” If the necessary triples aren’t in the table already, pick a length and determine whether it’s possible to find one Pythagorean triangle with a hypotenuse of the specified value and another Pythagorean triangle with a leg of that length. The method

**Problem:** A rectangular box is 12 inches long, 9 inches wide, and 8 inches deep. How far apart are the left, lower, front corner and the right, upper, back corner of the box?

**Problem:** Rewrite problem 2 so that $CD = 60$ and the resulting right triangle still has integer sides (you’ll need to specify $AB$).

**Pythagorean Triangles with Integer Altitudes**

Describe a method to create infinitely many Pythagorean triangles having altitudes with integer length.

**Problem:** Find 3 noncongruent triangles that have integer coordinates and sides. They don’t have to be right triangles.

**Problem:** Show that all triangles with side lengths given by the above formula do, in fact, contain a 60° angle. Between which two sides is the angle?

**Problem:** Find 3 noncongruent triangles that have integer sides and exactly one 60° angle.

**Problem:** Find 3 other boxes with dimensions satisfying the property that the three dimensions and diagonal are integer-valued.
you used in problem 11 in section 4 (to find triples with one entry equal to 12) will be a good start. Use it to create more examples (with different lengths).

12. In problem 1, you characterized those numbers that can be expressed as the sum of the squares of two counting numbers and also as the difference of the squares of two different counting numbers. Describe a method for generating infinitely many such numbers.

So many cousins, so little time. We hope that we have piqued your interest to explore further.