INTRODUCTION TO RESEARCH IN THE CLASSROOM: ANSWERS TO FREQUENTLY ASKED QUESTIONS

What is mathematics research?

Mathematics research is the long-term, open-ended exploration of a set of related mathematics questions whose answers connect to and build upon each other. Problems are open-ended because students continually come up with new questions to ask based on their observations. Additional characteristics of student research include:

- Students develop questions, approaches, and results, that are, at least for them, original products.
- Students use the same general methods used by research mathematicians. They work through cycles of data-gathering, visualization, abstraction, conjecturing, and proof.

- Students communicate mathematically: describing their thinking, writing definitions and conjectures, using symbols, justifying their conclusions, and reading mathematics.
- When the research involves a class or other group, the students become a community of mathematicians sharing and building on each other's questions, conjectures, and theorems.

How do students benefit from doing mathematics research?

Mathematics research influences student learning in a number of ways:

- Research provides students with an understanding of what it means to do mathematics and of mathematics as a living, growing field.
- Writing mathematics and problem-solving become central to student's learning.
- Students develop mastery of mathematics topics. Philosopher and educator **John Dewey** claimed that we don't learn the basics by studying the basics but by engaging in rich activities which require them. Research experiences require the repeated application of technical skills in the service of looking for patterns and testing conjectures (e.g., factoring and graphing polynomials for the Patterns in Polynomials project). It is this repetition, in the context of motivating and meaningful problems, that leads to greater understanding and retention of mathematics skills. During an investigation, students make connections between ideas that further enhance retention.
- Students develop their own mathematical aesthetic as they practice making choices about which aspects of a problem to investigate.
- Students develop both confidence as mathematical thinkers and enthusiasm to do more mathematics. The creativity, problem-solving, surprises, and accomplishments that are part of research help to answer students' questions about the value of studying mathematics. They are studying new methods so that they can answer their own questions. They are learning in order to do work that they care about at that moment (and not for a test or some far-off future task).
- Doing research is challenging and can be frustrating. Students' commitment to persistence and tolerance for frustration grow as they are supported, encouraged, and given repeated opportunity to think about and succeed with problems over days and weeks.

- Students learn to distinguish between different levels of evidence and to be skeptical in the face of anecdotal evidence. The habit of looking for counterexamples to claims is a core skill for critical thinkers in all aspects of life.

For which students is research appropriate?

This question is usually more bluntly framed as "Can kids really do this?!" The experience of teachers in all types of school settings is that *all* children can successfully engage in mathematics research. *Making Mathematics* teachers have undertaken research with urban, rural, and suburban students from grades 4 through 12. They have guided at-risk, honors, and English as a Second Language (ESL) classes through projects lasting from a few weeks up to a year. Students in math clubs, individual students, and home-schooled students have carried out successful investigations. One of our teachers first introduced research to her honors seventh graders. Once she was confident in her own experience, she tried the same project with two low-tracked eighth-grade sections. The quality of the questions, experimenting, reasoning, and writing was excellent in all three sections and indistinguishable between the honors and non-honors students. Research drew upon a richer array of student abilities than were assessed for tracking purposes.

Research can thrive in a heterogeneous class of students if you pick a project that does not require a lot of background to get started but which also inspires sophisticated questions. Students will pose problems at a level that is both challenging and appropriate for them.

How can I get my feet wet with research?

Making Mathematics teachers have been most comfortable trying research for the first time with one of their "stronger than average" sections. Some teachers have begun work with one or more interested students as part of a mathematics club or independent seminar. The purpose of these first excursions has been for the students to become familiar with the research process and for the teacher to see how students respond to lengthy, open-ended problem-solving.

Popular starting projects have been Trains, Connect the Dots, Hilgemeier's Likeness Sequence, and Raw Recruits. These projects are good starting points for any secondary group because they quickly inspire observations, conjectures, and new questions ("What if we do this…?") and can get to informal reasoning to justify some of the conjectures within a day or two. This easy entry is due to the familiarity of the content (e.g., counting, arithmetic, shapes).

You should commit at least three consecutive class periods at the start of a first investigation in order to maintain the momentum of the experience. You want students to appreciate that the questions are not typical quick exercises, so it is important that they get to wade into the work. Interruptions also make it harder for them to maintain a line of thinking. After the initial burst, you can sustain a project through weekly discussions of work done at home. If a problem is working well, do not be afraid to let kids pursue it for a long period of time. All of these projects have proven to remain challenging and interesting during weeks of student exploration (except for the Likeness Sequence, which works best as a shorter introductory activity for older students).

What can I do once my feet are wet?

If you have tried research with just a few students, try it with a class. If you have begun research with one class, try it with others. Read more chapters of the teacher handbook and integrate some of the supporting activities that focus on particular research skills. The most fun and greatest benefits accrue when research becomes an ongoing strand within a course. One investigation gives us a taste of research. When we engage in research regularly, we hone our intuitions about what approaches to attempt at each juncture in the process. Additionally, students who do research periodically start to apply to all of their mathematics studies the habits of extending questions, conjecturing, looking for patterns, generating confirming and counter-examples, and checking their reasoning carefully.

When students become really excited about doing mathematics and want to try a long-term project, you can form a seminar or club to support them as they work on one topic for a semester or more. Meetings can alternate between discussing the students' progress with their questions and studying specific research skills (e.g., proof techniques, writing definitions, etc.).

Problem-posing is central to long projects. Once a student has solved an initial question, they should look for extensions of the question that build on their work. They will discover that research problems can last forever. Each new piece of work can spawn many more questions for research. However, students need to be thoughtful about the research agenda that they pursue. Endless generalizations and extensions of a problem may not yield a satisfyingly cohesive research product. For example, the many cow problems listed in the problem-posing chapter are all related by context and type, but they may not produce some larger vision that makes the solving of the next cow problem easier. There may be no interesting *theory* of cow problems and ultimately one does not just want a bag of problems but a connected whole with overarching patterns and methods that recur throughout many of the questions and solutions.

What kind of support will I need?

Many teachers independently introduce research into a class. Your work will have greater impact on students if they encounter research in all of their mathematics classes. Both for that reason and in order to feel less isolated as you experiment, it is helpful to recruit one or more colleagues to try out research along with you. Share ideas and observations and even visit each other's classes on days when the students are doing research. Talk with your department head or supervisor to garner support for your efforts.

If you want an advisor for yourself or an outside audience for the work that your students do, you can contact the mathematics or mathematics education department at a local college and ask if any of the professors would be willing to serve as a mentor (either via email, phone, or in person) for you and your class. We have also found good mentors contacting corporations that employ scientists and mathematicians. Your mentor may just communicate with you or she may be willing to read updates or reports from the students and provide responses. You should make these exchanges via your email account—parental consent is required by law for direct internet communication. Be sure to let any prospective mentor know what your goals and expectations are for the students and for their involvement.

Mentors can help in a number of ways. They can:

- Help pick an area of inquiry and establish goals.
- Make suggestions about next steps (e.g., when to apply a research technique or content area, whether to redirect a class's efforts).
- Ask clarifying questions about the student's mathematical statements.
- Help students learn how to prove their claims.
- Study and reflect upon student work with you.
- Provide an authentic outside audience for student efforts.
- Provide emotional support such as encouragement, perspective, and advice.
- Identify resources.

What do I need to do before I begin?

- If you have never done any mathematics research yourself, it is time to join in! Find a colleague, pick a *Making Mathematics* project, and start your work looking for patterns, trying to state clear conjectures, searching for proofs or disproofs, and studying new, related problems (read about the Research Process in the Introductory Explorations chapter and work through the Conjectures and Proof chapters together as well). Many *Making Mathematics* teachers have found the summer a good time for professional growth via a research project.
- Your commitment to doing research will be clearer and more resilient if you first list your goals for your students and decide what role research plays in reaching those goals (see Setting and Sharing Goals). If you come to feel that research is a necessary outcome of studying mathematics, then your questions will shift from "*can* I do this?" to "*how* can I do this?"
- Decide what your goals for your initial foray are (e.g., each group will be responsible for posing and resolving one question or students will each have their own conjecture) and how much time you plan to spend on the project.
- Pick a project topic.
- Since research is unfamiliar to many parents and administrators, you may want to anticipate any questions that will arise by discussing your plans ahead of time. You can send a letter home to parents that helps them to understand what you will be doing and why. You or your department head can talk with your principal about your goals for your students.

How do I choose a project topic?

Choose projects that are at the right level of challenge for your students. For novice student researchers, it is preferable if the focus is on learning about the research process. Projects that involve familiar content allow for a gentle introduction and for the greatest possibility of multiple interpretations and avenues of exploration that draw upon well-developed student understandings. When students can jump in fast, they are more likely to work through the research cycle more than once and grasp the iterative and open-ended nature of research. We describe these projects as having a low threshold and a high ceiling—every student can participate and there is lots of room for the most advanced students to find challenging questions.

As students gain experience with research, they will be more confident and ready to tackle questions involving less familiar areas of mathematics. It is at this point that it will be easier to have students learn new mathematics topics in the context of research. This combination will allow you to give students practice developing important mathematical habits of mind while covering the content required of a given course (see How can I balance the development of research skills with the need to cover specific mathematics topics? below).

Certain projects are particularly inspiring for students because of their visual appeal. For example, the pictures that emerge during the Patterns in Pascal's Triangle or Inspi investigations can catch students' attention and stimulate them to look for the underlying explanations of what they see. See Alan Schoenfeld's discussion of criteria for good problems at http://www-gse.berkeley.edu/Faculty/aschoenfeld/WhatDoWeKnow/What_Do_we_know02.html).

You need to consider your own comfort level when picking a project as well. You may want to spend some time working on and familiarizing yourself with the questions before you introduce them to the class. Do not feel that you have to have the entire project mastered. Once students get working, they invariably raise questions that none of us anticipate, so it is impossible to figure out all of the answers ahead of time (see What if I am not familiar with a problem? below and Being a Role Model in Getting Stuck, Getting Unstuck).

If you are working with a small number of students, you may want to have them pick the project. One advantage to giving students a choice is that they will feel more motivated having picked a question that most interests them. They will also see that you want them to develop their own personal mathematical tastes. It is better if at least two or three students work on a given project so that they can share ideas with each other. We have, however, seen many cases of individual students working productively on problems that they have chosen or posed themselves.

Finally, one or more students may come to you with an original question or you can invite students to pose their own questions (see Problem Posing). Students who tackle their own questions are coming into their own as mathematicians, but there is a caveat that accompanies such an endeavor. Since the problems are original, it may not be clear ahead of time if they are too difficult for the student. Similarly, the examples may not turn out to follow any recognizable patterns or yield any conjectures. Original questions do not come with guarantees.

What if I am not familiar with a problem?

Perhaps the greatest anxiety that teachers express about doing research is that they themselves may not be able to answer the questions that students are exploring. As noted above, we cannot expect to know all of the answers to all questions, nor should we portray ourselves in that light. It is not our job to answer all of questions that students might pose—it is our job to model for them the questions that they should be asking themselves when they are having difficulty making progress (see Getting Stuck, Getting Unstuck). We have, in fact, been unable to answer numerous problems posed by our researching students, in part because they have had much more time to think about each question than we have and in part because some have been quite hard (and remain unsolved). Consider the following note from a mentor to a teacher who had just finished a research unit with her class:

> So much of the stuff you write about—"they no longer needed me to validate their work... they saw how much more complicated the problem became..."—that's all so important in understanding how mathematics really works. If you're trying to work on a problem that's new, no one *can* validate you're work because you're the first one to try it! *You* have to figure it out, convince yourself, and then convince others.

For every project that a class investigates, the students should have a running list of conjectures that they have not yet proven or disproven. This will help them see that it is the natural state of mathematics to have open questions with which many researchers are grappling.

How do I help my students during research?

When students are engaged in research, our job is to teach them the stages of the process and to coach them to develop the habits that lead to success. The most common coaching maneuver is to ask a question. The purpose of an inquiry is to model the types of questions that the student should be asking herself and to help the student and her teacher understand what she is doing and why.

The other key to helping your students is to be enthusiastic about their ideas and questions and to be patient when they are stuck. Acknowledge both the satisfactions and the difficulties of research so that students can address the emotions that accompany learning. Because progress in research can take time and come sporadically, it is important that you remove any external stresses when students begin research (unless you are very careful, grading can be a distraction and hindrance for novice researchers). Here are some of the basic acts that teachers use when coaching students (note that many of these are just statements of good teaching in general):

1. If a student's language is unclear, ask for clarification. Offer edits when appropriate.
2. When they exhibit a research skill or approach that has general utility, offer some celebratory language letting them know that they did something important. Then, encourage them to note what they did as a technique for future use.
3. If they have a single instance, ask what it might be an instance of, if they can generalize it, and in what direction. For example, when they describe how many ways 2 recruits out of 6 can face the wrong way, ask them what happens if they change the 2 or the 6. Encourage them to look for patterns.
4. Whether their analysis of a problem is incomplete or complete, ask them how they know that they have been systematic or rigorous. Help students find their own errors rather than pointing them out directly. Rather than serving as the certifier of results, ask questions ("Why do you think that?" or "How could you get that same answer in a different way?"). Ask regardless of the correctness of a claim so that students develop the habit of always checking their thinking. Have peers check each other's work and provide independent reasons for their support or concern.
5. When kids make an interesting observation, note their creativity and ask why their conjecture might be true. Encourage them to seek justifications that convince us that the patterns are genuine and will continue. In short, guide students to prove their claims.
6. As they look for explanations, help them distinguish between examples and reasons.
7. Suggest natural next steps to help keep students from stalling. If they are stuck because they lack necessary background, you can teach the ideas to them or point them to a resource that they can use to learn the ideas on their own.
8. Summarize and organize the different things a group has said in order to give them an overall picture and direct them to the next stage of their work. This step is somewhat more hands-on than is always necessary. Sometimes a mentor asks the students to organize and re-write their efforts to date. In cases where time is limited, this option may make sense.
9. If a student has a satisfactory conjecture plus explanation (i.e., a theorem), ask what further generalization or change she might explore.

10. Once groups have made some progress, get them sharing questions, insights, and findings to help build a mathematical community out of different parallel (and sometimes overlapping) investigations.
11. Help kids make connections within areas of their investigation as well as to mathematical questions that they are not exploring or may not have heard about.
12. Help decide when it is time for a research experience to wind down or wrap up.

See the Appendix A for mentor comments that exemplify the above list of responses.

How should I use the warm-up problems?

Each Making Mathematics project has associated warm-up problems. Which, if any, you use will depend on the background of your students. Students can start most research projects at an interesting level without work on any of the warm-up problems. In some cases, you may want to use the warm-ups after an initial exploration so that students are thinking about the problems within the context of the main project questions. Certain warm-up problems may turn out to be lengthy research challenges themselves (so gauge your available time accordingly or just use the warm-up as a research question).

What might a research sequence within a class look like?

The teaching notes accompanying the Connect the Dots project and Introductory Explorations activity can serve as models that you can adapt to other projects. As noted earlier, it is best if you can introduce research with a burst that permits a coherent presentation of the research process before separating discussions with several days of non-research studies.

Once research is underway, each student or group of students may work on different, but related, questions. During whole-class discussion, classmates should describe the different problems that they are exploring. Students should report back on their progress (new questions, conjectures, proofs, etc.) periodically.

At the end of a class session devoted to research, each group should give themselves a homework assignment in their logbooks. You can check these recorded tasks to make sure that the assignments were meaningful and check the subsequent entry in the logbook to make sure that the student made reasonable progress with the tasks. Typical homework challenges include:

- Extend a pattern, generate more data.
- Try to prove a particular conjecture.

- Test a bunch of conjectures with different cases to see whether counterexamples can be found.
- Try to find a formula or rule for a pattern.
- Identify and learn about areas of mathematics that might be helpful to the investigation.
- Read about related problems and how they were solved.
- Pose extensions of the project.

Students can think about where they are in the research process in order to decide what step to attempt next. Their work should have some narrative explanations ("I did this because…"). Students can work on their homework for a few days, but groups will also need regular class time to catch up on each other's thinking, to work together, and to then coordinate next steps before their next stretch of independent work.

Although the teaching notes for many of the Making Mathematics projects suggest what to do on the first day, the second day, and so forth, you will need to pace the phases of a particular investigation according to the length of your class periods and the timing of a given class's particular questions and discoveries. Here are some other decisions that you should be alert to as work proceeds:

- Students will naturally exhibit important research skills such as posing a conjecture, organizing data in an effective manner, or inventing a new definition. When this happens, you want to identify the skill and discuss its importance to research. For example, a student might note the existence of a counter-example to a classmate's conjecture. If students do not already know about counter-examples, you could stop to highlight the contribution and do a side lesson (with help available from the teacher handbook) on examples (generating test cases, remaining skeptical in the face of confirming examples, extreme and degenerate cases, and counter-examples).
- Issues will arise unpredictably and student comments may simultaneously pull a class in a number of possible directions. For example, one student may invoke a counter-example for the first time and another might pose two new questions to explore. Where do you head first? You should try to strike a balance between developing a formal understanding of research skills and allowing the research process to unfold without too many interruptions. It is always good to at least give a name to a new research skill or problem-solving strategy when a

student demonstrates it. You can then return to a discussion of the considerations associated with that habit in depth at a later time.

- As an investigation continues, the difficulty of generating further examples may become an impediment to further progress. You should help the students decide whether carrying out the steps needed to find new examples is still itself illuminating or whether just the data gleaned from the examples is what they need. If the latter is the case and there is a way to use technology to speed up the work, it might be worth taking the time to teach the class or a particular group how to use the appropriate tool. For example, they might benefit from assistance doing symbolic manipulations using a computer algebra system (CAS), geometric constructions using a dynamic geometry program, or finding numeric examples using a spreadsheet.

As a class works thorough its early research experiences, be sure to document for them as much of their work as possible. Posters listing the students' conjectures, questions, and theorems help students grasp the cyclical nature of the research process. They see how their different questions connect and build upon each other and learn which research methods are most helpful at which stages of an investigation. After these beginning projects, students are ready to work more independently and should be encouraged to pose their own questions for research.

Stand-alone activities from the teacher handbook and mathematics tools entries can be used during research explorations or in between as a way to keep research thinking fresh when other topics are taking central stage in your class. When used in the midst of an investigation, they are a response to a "teachable moment" that makes them a timely interruption. You can also intersperse readings (see the resources chapter) about present-day mathematicians and their work as a way to broaden students' view of the field and to inspire them with the personal stories of persistence and discovery.

See Writing Math Research Papers by Robert Gerver for more advice on structuring individual research projects.

How does a research project end?

A project can end when a student or group has resolved some central question. Often, there are many questions and, after good progress with some of them, students' enthusiasm for the others may wane. You may have established certain goals for students: to create a proof, to generate a few clear conjectures, to pose a new problem and make progress with it. Each of these possibilities is a reasonable time for work on a project to end. Students can come to a satisfying sense of closure even with a project that leaves many unanswered questions. That feeling can be enhanced if they write a final report that summarizes their main questions and work and that concludes with a list of possible extensions worth exploring. See Presenting Your Research for ideas about formal write-ups for students who have engaged in a lengthy examination of a research question.

How will doing research affect my workload?

Ultimately, research is no more demanding on your time than teaching that is more traditional. In some cases, it shifts the balance so that you spend less time preparing lessons and more time responding to student work. If you have not taught research before, there will be an initial need to think through the different issues that will arise in class. This work will prepare you to take advantage of any “teachable moments” (student comments that can lead the class to new understandings). The teacher handbook is a valuable resource as you develop experience doing research with students.

One strategy for managing the demands of teaching research is to keep good notes on your observations during class. Thorough ongoing documentation will facilitate the comments that you need to make when you collect work because you will have a good sense of the entire research process that an individual or group has gone through. The more often you can read and respond to student’s entries in a their logbooks, the better, but you do not have to collect everyone’s work all at once. You can sample a few each night. Lastly, having each group submit a single final report reduces the number of papers that you need to study to a manageable number.

How can I balance the development of research skills with the need to cover specific mathematics topics?

> Teacher: Since I have a curriculum I have to cover, which really takes the whole school year, how do I cover that curriculum while implementing those problems that deepen the understanding of the required curriculum and is there some way to perhaps replace chunks of my text with such problems? I find balancing what I WANT to do and what I MUST do very difficult.
>
> Mentor: I appreciate your frustration about the tension between covering technical content and giving your students the opportunity to learn about the process of doing mathematics. There

> is no question that teachers are being asked to whiz through too many topics. I try to remind teachers of what they already know: when we go too quickly, the material is not mastered well and so we are not being efficient.

The above exchange between a Making Mathematics teacher and her mentor is typical of the most common and emotional question with which teachers interested in research have grappled. Many have expressed stress at feeling trapped by competing demands. In some cases, the answer is simple: if there is a major state test next week and you need to cover five topics, it is definitely a bad time to start research. But, if you are months away and you consider how often students forget what they have studied, now is a good time to introduce your students to mathematics investigations.

> "There is, of course, a cost to having the students engage more deeply in the mathematics: one "covers" less. However, when the payoffs include much deeper understanding, much longer retention of the content, enthusiasm, and the fact that the students get a much better sense of the mathematical enterprise, the price in (ostensible) coverage is a small one to pay." – Mathematics Educator Alan Schoenfeld.

As Schoenfeld and Dewey remind us, the content versus research question reflects a false dichotomy. We know how fruitless it is to teach disconnected topics. If you do not use knowledge in active ways that allow you to make meaning of what you have learned, you do not retain that learning. Why do students seem to forget so much of what they study? Sometimes, they still have the skills but are only able to apply them when prompted (e.g., "I am doing a chapter four problem" or "I was told to use triangle trigonometry techniques"). Sometimes, the learning experience was not memorable (consider what you have remembered and forgotten from high school and try to identify why). The more research work becomes a strand throughout a course and a school's curriculum, the better the interconnections between, and mastery of, technical content will be.

The NCTM Standards include many important goals (e.g., being able to conjecture, show persistence in problem solving, develop mathematical models, etc.) that we are supposed to "cover" that do not fit well in the framework of timed tests.

So, how do we combine research and technical content goals and what are some of the challenges that we face in our efforts? We can choose a research problem that will reinforce technical skills that a class has already studied. Alternatively, we can pick a problem that will introduce our students to and help them develop an understanding of a new topic. For example, we could use the Game of Set research project in place of or after a textbook introduction on combinatorics.

One problem that arises when using a research experience as a way to develop or reinforce a particular technical skill is that students' questions and methods may not head in the direction that you expected. One group of students, presented with the Raw Recruits project, wanted to be able to test the behavior of all starting positions. To do so, they had to know how many starting positions there were and so, unwittingly, began a combinatorics exploration of the possible arrangements involving *n* recruits with 2 facing the wrong way. Another group created a circular version of the problem and learned about periodic behavior. If you tell students to use a particular technique, then you short-circuit the research process. You are also risking turning the effort into a planned discovery activity, which usually lacks the motivational and intellectual power of true research.

You can address this problem in a few ways. A careful choice of project or framing of the question can often make certain skills inevitable. For example, a high school class proving theorems about Pythagorean Triples would be hard pressed to avoid using algebraic expressions or thinking about factors. You can also add your own questions to the class's list. This makes you a participant in the process and assures that the class will spend some time on the issues that you want considered. Alternatively, you can let the students' work take them where it will knowing that some other important area of mathematics is being developed or reinforced that you will not have to spend as much time on in the future. Then, after the research is over, you can return to the topic that you originally had in mind.

When students do get to follow their own intellectual muse, they are more likely to experience a wide range of mathematics topics. For example, in a class of fifth graders working on the Connect the Dots project, one student asked what would happen if each jump was chosen randomly. The shapes were no longer as attractive, but the question of whether they would ever close led to the idea of expected value. An independent research project on randomness in DNA led a student to study matrices and Markov processes. Students will teach themselves a chapter of content from a textbook if they think it will help them on a task about which they care.

How should students keep track of their work?

> Writing is an exploration. You start from nothing and learn as you go. – E. L. Doctorow

Students should maintain a logbook throughout a research experience. In this logbook, they will keep a record of everything they do and everything they read. Students should be encouraged to write down questions that they have when they are reading or working on their mathematics. This journal will become a record of the student's entire mathematics research experience. It will be an invaluable tool during their investigation and as they produce their final write-up at the end of the project.

There are two common approaches to the organization of a mathematics logbook. You should decide which type of logbook better meets the needs of you and your students.

For lengthy research projects, some teachers prefer that students use a bound logbook. Science logbooks, filled with graph paper and pre-numbered pages, are ideal for this sort of journal. Since the page numbers come pre-printed, it is obvious that something is missing if a page is torn out. Logbooks of this type encourage students to keep all of their work, even work that they do not actually use in their final project. It demonstrates a clear progression of mathematical development and thought throughout the research experience. If students want to add copies of articles or diagrams, they can staple or tape them into place. A formal logbook of this type is often *required* for science fair projects. See Appendix B for student instructions for this type of logbook.

In other cases, we recommend the use of loose-leaf binders for logbooks. Loose-leaf notebooks make it easier to keep material in sections and to move pages around. They also make it easier for teachers to ask students to hand in portions of their logbook because they can remove the pages and then put them back when the teacher is done looking at them. Students can insert computer printouts, pictures, copies of articles, etc. in an appropriate place. (Gerver, pp. 91-92). See Appendix C for student instructions for this type of logbook.

No matter which format is used, we recommend that students:

- Date their entries.
- Always work in pen. This forces them to…
- Not erase mistakes. Instead, they should cross out errors with a single line, thus making it possible to recover ideas that turn out to have worth.

- Write down a full citation for every book, article, or web site referenced. This information should be recorded at the start of the section of notes from that source (i.e., before the student begins to read and take notes).
- Show all reasoning and work (including calculations and graphs done with calculators or computers) and not just "final" answers.

Students should write what they are feeling and thinking in their logs. The log is a record of a student's dialogue with herself and the mathematics ideas of her project. Dry, formal writing is an impediment at this stage of work. One of our students had the following observations and questions in his log:

- Wait! Do negative integers work for *a*?
- Hmm. Repeating and it's a whole number imaginary.
- Back to my earlier question, can I create an $a + bi$ (a, b not 0) irrational fractional base like the others?
- A sudden revelation: We know why rational numbers and not just whole numbers are…
- Doesn't seem like much here, so maybe I'll try some of Bergman's division algorithms and see if I can get some insight.

His comments served to provide a clear narrative of his reasoning and motivation.

Neatness and organization are not an intrinsic virtue in a log book, but they are important to the extent that the student must be able to make sense of her writing days later and will not want messiness to distract any reader of her log.

When and how should students work in groups?

Students benefit from group work in a number of different ways. Students can more readily adjust to the unfamiliar aspects of research with the support and exchange of ideas that a group can provide. Group efforts allow students to contribute their strengths to a research project without getting stuck because of an area of weakness. In other words, groups can be crucial to the early confidence-building stages of teaching research. As research continues in a class, group efforts allow students to discover the power of being part of a mathematical community that is building an interconnected set of mathematics ideas stimulated by each other's thoughts and questions.

Although a whole class can work on a problem together, smaller groups are preferable inasmuch as they give more students the chance to participate. Multiple groups are also more likely to produce an interesting variety of ideas than will a whole-class discussion. Before starting students off in groups for an extended activity (doing research or anything else), it is worthwhile presenting the discussion questions from the Building Collaborative Skills chapter.

We recommend giving each student the chance to spend some time individually making sense of a problem before putting groups together. This initial period allows students to figure out at their own pace what they know about a problem and what questions they have. After the class makes a list of their questions, you can form groups and ask each one to pick a question for their members to explore. Alternatively, you can invite students to join a group based on which question they would like to explore ("If you like problem A, please move over here."). Although there is no hard and fast rule for group size, groups of three or four students often provide a good critical mass of ideas while allowing for plenty of participation.

You should decide whether you want each group to appoint a daily recorder who writes down a full description of all of the group's work in a log or whether each member is responsible for keeping a record. If students are going to be working at home on the problems, the latter arrangement may be best (although in some classes the teacher photocopies the notes at the end of class for each group member).

When groups work in class, your job is to visit each group, to observe and take notes, and to ask questions. Your goal is to assess where the students are heading (e.g., by asking "What are you all working on at this moment?" followed by "How does that relate to the main question that you are investigating?") and whether they can explain their own decision-making and reasoning (e.g., "Why do you think that that conjecture might be true?"). See How do I help my students during research? and Getting Stuck, Getting Unstuck for more advice on helping groups during the research process.

Students also grow from doing research independently. Independent work allows them to follow their own muse, to make progress at their own pace, and to work through challenges and learn from that process in all of its richness and difficulty. The victories are all their own.

What role can technology play in research?

Advanced calculators and computer software can promote research because, in the exploration of functions, numbers, and shapes, they can change the nature and number of questions that students ask. It can be quite exciting when students take advantage of technology's ability to facilitate rote work and expedite deeper conjecturing about patterns in mathematics.

For example, a student might look at how $x^n - 1$ factors for different whole numbers n using a computer algebra system (CAS) such as Mathematica or the TI-92. But, they are unlikely to be willing to factor $x^{18} - 1$ without computer help any more than we would be likely to do long division of 6-digit numbers. The field of fractals and chaos would not have blossomed without the aid of computers that freed researchers up to ask questions that would have been unanswerable in the past. Many of these questions only yielded to analysis after simulations and number crunching revealed patterns. Similarly, access to a spreadsheet or dynamic geometry program can free students to ask "What if…?" about mathematical objects that would be too daunting to study without a technological boost.

As with any tool, students need to learn the benefits and limitations associated with using a particular piece of software. For example, if a student working on a difficult combinatorics problem writes a program to "number crunch" an answer instead of patiently analyzing the structure of the situation, she will usually fail to develop a solution that she can generalize. She is likely to miss the insight that a pencil-and-paper route might have provided.

Although CAS programs can produce exact answers to many problems, most calculators and programs still display approximations, such as 1.7320508 instead of $\sqrt{3}$. Frequently, numbers written in radical or fraction form are more memorable and yield more readily recognizable patterns than their decimal equivalents. Students using mathematical software should be able to recognize important numbers in different forms.

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APPENDIX A

Sample Responses to Middle School Groups Working on the Raw Recruits Project (taken from email exchanges between students and a Making Mathematics mentor.

1) Ask for clarification

   Here, I think that I do understand the problem, but again don't quite understand what you're saying about the example. You say "this is the one," but I don't know which one you are referring to. What rule does a "wrong" recruit follow? And what do you mean by "it goes around then it goes back the other way around"? Possibly color would help identify where people should look or…?

2) Identify and celebrate research skills

   a) This is really a lovely piece of work. You're right. I was slightly confused by the rectangles, but I see it all now. Your final explanation has a couple of parts that I especially love: the exception (six) that you call attention to, and the symmetry (when you get past three errors…). In mathematics, we sometimes use the term "degenerate case" (slightly humorous, I guess!) to talk about the six-error case (or zero-errors, for that matter) because, even though they're exceptions to the rule, they don't really cause any trouble because "of course they stop – there's nothing to get them started." The symmetry is a really big deal mathematical idea that's used all over the place and often startlingly hard for people to get, so congratulations! And thanks for sharing your (great!) thinking with me. I really enjoyed it, and hope you did, too.

   b) Wow! Mathematicians really like to use arguments like this! This is called "reasoning by symmetry," because you're really saying that <<>>>> is so much like >><<<< that you can make the same claims about both of them.

   Nice! How about taking it just one step further and seeing if you can explain why, as you put it, "The mistakes always end up the same, only with the mistakes facing the opposite direction as the people who are facing the right way." Being able to say clearly why that must happen would make an excellent next step for you.

   c) When students came up with an effective representation of the problem: Your numerical way of describing the arrangements is not something I've seen before, and seems like an excellent way of checking to see whether all the possibilities have been found without duplication. (Yeah, I know there was a slip-up this time, but I bet it's a more reliable system than the >><><< system, in general!) I would love to see you develop this system and show how you could use it to predict the number of arrangements for other length lines. That would be something quite new and, I think, interesting!

d) I know I'm supposed to comment mostly about the math, but I just had to say how really nice it is to read something that is so clearly written, and in a style that makes it feel directed to me. I spend a great deal of time teaching the people who work with me to write in this kind of direct and clear style. Yay!

e) NICE! This is a classic example of things that mathematicians look for: something that doesn't change, while other things around it do change. For example, in a table like

x	y
3	7
4	9
5	11

all the numbers are changing, but what doesn't change is the relationship between x and y: y is always one more than twice x. That is, $y=2x+1$. Finding what doesn't change "tames" the situation. So, you have tamed this problem! Yay. And if you want a fancy mathematical name for things that don't vary, we call these things "invariants." The number of messed-up recruits is invariant, even though they are all wiggling back and forth, trying to figure out which way is right!

3) Encourage generalizations

So, of course, the next question that comes to my mind is how to generalize what you've already discovered: there are 15 ways that 2 mistakes can be arranged in a line of 6 recruits. What about a different number of mistakes? Or a different number of recruits? Is there some way to predict? Or, alternatively, is there some way to predict how these 15 ways of making mistakes will play out as the recruits try to settle themselves down?
Which direction interests you?

4) Inquire about reasoning and rigor

The students were looking at the number of ways the recruits could line up with 2 out of *n* faced the wrong way: Anyway, I had a question of my own. It looks like the number of possibilities increases pretty fast, as the number of recruits increases. For example, I counted 15 possibilities in your last set (the line of six). What I wonder is this: when the numbers get that large, how you can possibly know that you've *found* all the possibilities? (For example, I noticed that >>>><< is missing.) The question "How do I know I've counted 'em all?" is actually quite a big deal in mathematics, as mathematicians are often called upon to find ways of counting things that nobody has ever listed (exactly like the example you are working on).

The students responded by finding a pattern for generating the lineups in a meaningful order: The way that we can prove that we have all the possibilities is that we can just add the number of places that the second wrong person could be in. For example, if 2 are wrong in a line of 6, then the first one doesn't move and you count the space in which the second one can move in. So for the line of six, it would be 5+4+3+2+1=15. That is the way to make sure that we have all the ways. Thanks so much for giving challenges. We enjoyed thinking!

5) Work towards proof

a) The group wrote the following: When we found out that 6 recruits had 15 different starting arrangements, we needed more information. We needed to figure out how many starting positions are there for a different number of recruits.

By drawing out the arrangements for 5 recruits and 7 recruits we found out that the number of starting arrangements for the recruit number before plus that recruit number before it would equal the number of starting arrangements for that number of recruits.

We also found out that if you divide the starting arrangements by the number of recruits there is a pattern.

6/4 = 1.5

10/5 = 2

15/6 = 2.5

21/7 = 3

28/8 = 3.5

To which the mentor replied: Wow! I don't think (in all the years I've been hanging around mathematics) I've ever seen anyone describe this particular pattern before! Really nice! If you already knew me, you'd be able to predict what I'm about to ask, but you don't, so I have to ask it: "But why?" That is, why is this pattern (the 6, 10, 15, 21, 28…) the pattern that you find for this circumstance (two recruits wrong in lines of lengths, 4, 5, 6, 7, 8…)? Answering that—explaining why you should get those numbers and why the pattern must continue for longer lines—is doing the kind of thing that mathematics is really about.

b) Responding to students studying a circular variation of raw recruits that never settled down: This is a really interesting conclusion! How can you show that it will always continue forever and that it doesn't matter what the original arrangement was? Have you got a reason or did you try all the cases or…? I look forward to hearing more from you.

6) Distinguish between examples and reasons

a) You have very thoroughly dealt with finding the answer to the problem you posed—it really does seem, as you put it, "safe to say" how many there will be. Is there a way that you can show that that pattern must continue? I guess I'd look for some reason why adding the new recruit adds exactly the number of additional cases that you predict. If you could say how the addition of one new recruit depends on how long the line already is, you'd have a complete proof. Want to give that a try?

b) A student, working on Amida Kuji and having provided an example, wrote the following as part of a proof: In like manner, to be given each relationship of objects in an arrangement, you can generate the arrangement itself, for no two different arrangements can have the same object relationships. The mentor response points out the gap and offers ways to structure the process of extrapolating from the specific to the general: This statement is the same as your conjecture, but this is not a proof. You repeat your claim and suggest that the example serves as a model for a proof. If that is so, it is up to you to make the connections explicit. How might you prove that a set of ordered pairs, one per pair of objects forces a unique arrangement for the entire list? Try thinking about a given object (e.g., C) and what each of its ordered pairs tells us? Try to generalize from your example. What must be true for the set of ordered pairs? Are all sets of nC2 ordered pairs legal? How many sets of nC2 ordered pairs are there? Do they all lead to a particular arrangement? Your answers to these questions should help you work toward a proof of your conjecture.

9) Encourage extensions

What you've done—finding the pattern, but far more important, finding the explanation (and stating it so clearly)—is really great! (Perhaps I should say "finding and stating explanations like this is real mathematics"!) Yet it almost sounded as if you put it down at the very end, when you concluded "making our project mostly an interesting coincidence." This is a truly nice piece of work!

The question, now, is "What next?" You really have completely solved the problem you set out to solve: found the answer, and proved that you're right!

I began looking back at the examples you gave, and noticed patterns in them that I had never seen before. At first, I started coloring parts red, because they just "stuck out" as noticeable and I wanted to see them better. Then, it occurred to me that I was coloring the recruits that were back-to-back, and that maybe I should be paying attention to the ones who were facing each other, as they were "where the action was," so I started coloring them pink. (In one case, I recopied your example to do the pinks.) To be honest, I'm not sure what I'm looking for, but there was such a clear pattern of the "action spot" moving around that I thought it might tell me something new. Anything come to your minds?

10) Build a Mathematical Community

I just went back to another paper and then came back to yours to look again. There's another pattern in the table. Add the recruits and the corresponding starting arrangements (for example, add 6 and 15) and you get the next number of starting arrangements. I don't know whether this, or your 1.5, 2, 2.5, 3, 3.5… pattern will help you find out why 6, 10, 15… make sense as answers, but they might. Maybe you can work with [your classmates] who made the other observation to try to develop a complete understanding of the problem.

11) Highlight Connections

Your rule—the (n-1)+(n-2)+(n-3)+… +3+2+1 part—is interesting all by itself, as it counts the number of dots in a triangle of dots. See how?

12) Wrap Up

This is really a very nice and complete piece of work: you've stated a problem, found a solution, and given a proof (complete explanation of why that solution must be correct). To wrap it up and give it the polish of a good piece of mathematical research, I'd suggest two things.

The first thing is to extend the idea to account for all but two mistakes and the (slightly trivial) one mistake and all but one mistake. (If you felt like looking at 3 and all but 3, that'd be nice, too, but it's more work—though not a ton—and the ones that I suggested are really not more work.)

The second thing I'd suggest is to write it all up in a way that would be understandable by someone who did not know the problem or your class: clear statement of the problem, the solution, what you did to get the solution, and the proof.

I look forward to seeing your masterpiece!

APPENDIX B

Advice for Keeping a Formal Mathematics Research Logbook

As part of your mathematics research experience, you will keep a mathematics research logbook. In this logbook, keep a record of everything you do and everything you read that relates to this work. Write down questions that you have as you are reading or working on the project. Experiment. Make conjectures. Try to prove your conjectures. Your journal will become a record of your entire mathematics research experience. Don't worry if your writing is not always perfect. Often journal pages look rough, with notes to yourself, false starts, and partial solutions. However, be sure that you can read your own notes later and try to organize your writing in ways that will facilitate your thinking. Your logbook will serve as a record of where you are in your work at any moment and will be an invaluable tool when you write reports about your research.

Ideally, your mathematics research logbook should have pre-numbered pages. You can often find numbered graph paper science logs at office supply stores. If you can not find a notebook that has the pages already numbered, then the first thing you should do is go through the entire book putting numbers on each page using pen.

- Date each entry.
- Work in pen.
- Don't erase or white out mistakes. Instead, draw a single line through what you would like ignored. There are many reasons for using this approach:
  - Your notebook will look a lot nicer if it doesn't have scribbled messes in it.
  - You can still see what you wrote at a later date if you decide that it wasn't a mistake after all.
  - It is sometimes useful to be able to go back and see where you ran into difficulties.
  - You'll be able to go back and see if you already tried something so you won't spend time trying that same approach again if it didn't work.
- When you do research using existing sources, be sure to list the bibliographic information at the start of each section of notes you take. It is a lot easier to write down the citation while it is in front of you than it is to try to find it at a later date.
- Never tear a page out of your notebook. The idea is to keep a record of everything you have done. One reason for pre-numbering the pages is to show that nothing has been removed.
- If you find an interesting article or picture that you would like to include in your notebook, you can staple or tape it onto a page.

APPENDIX C

Advice for Keeping a Loose-Leaf Mathematics Research Logbook

As part of your mathematics research experience, you will keep a mathematics research logbook. In this logbook, keep a record of everything you do and everything you read that relates to this work. Write down questions that you have as you are reading or working on the project. Experiment. Make conjectures. Try to prove your conjectures. Your journal will become a record of your entire mathematics research experience. Don't worry if your writing is not always perfect. Often journal pages look rough, with notes to yourself, false starts, and partial solutions. However, be sure that you can read your own notes later and try to organize your writing in ways that will facilitate your thinking. Your logbook will serve as a record of where you are in your work at any moment and will be an invaluable tool when you write reports about your research.

Get yourself a good loose-leaf binder, some lined paper for notes, some graph paper for graphs and some blank paper for pictures and diagrams. Be sure to keep everything that is related to your project in your binder.

- Date each entry.
- Work in pen.
- Don't erase or white out mistakes. Instead, draw a single line through what you would like ignored. There are many reasons for using this approach:
  - Your notebook will look a lot nicer if it does not have scribbled messes in it.
  - You can still see what you wrote at a later date if you decide that it wasn't a mistake after all.
  - It is sometimes useful to be able to go back and see where you ran into difficulties.
  - You'll be able to go back and see if you already tried something so you won't spend time trying that same approach again if it didn't work.
- When you do research using existing sources, be sure to list the bibliographic information at the start of each section of notes you take. It is a lot easier to write down the citation while it is in front of you than it is to try to find it at a later date.
- Be sure to keep everything related to your project. The idea is to keep a record of everything you have done.
- If you find an interesting article or picture that you would like to include in your notebook, punch holes in it and insert it in an appropriate section in your binder.