Necessary Condition and Sufficient Condition by Shuei Sasaki

in

Becker, Jerry & Shimada, Shigeru (1997). *The open-ended approach: a new proposal for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

NECESSARY CONDITION AND SUFFICIENT CONDITION

Shūei Sasaki
Yamagata Minami Upper Secondary School, Yamagata

The Problem and Its Context

The problem

1. At the end of each of the following statements, write "T" if the statement is true, and "F" if it is false.

   A. If $x = 2$, then $x^2 = 4$.
   B. If $x = 2$, then $x^2 - 4x + 4 = 0$.
   C. If $x = 2$, then $x^2 - 2x + 1 = 0$.
   D. If x is a divisor of 3, then x is a divisor of 12.
   E. If x is a divisor of 6, then x is a divisor of 12.
   F. If x is a divisor of 9, then x is a divisor of 12.
   G. If one of the pair (x, y) is positive, and the other negative, then xy is negative.
   H. If xy is negative, then one of the pair (x, y) is positive and the other negative.
   I. If both x and y are positive, then xy is positive.
   J. If xy is positive, then both x and y are positive.

2. Fill in the following blank with an expression about x or y that fits it. Write as many expressions as you can.

   If both x and y are even, then ________ is even.

3. Fill in the blank with an expression about x or y that fits it. Write as many expressions as you can.
   A. If either x or y is odd and the other even, then _________ is odd.
   B. If _________ is odd, then either x or y is odd and the other even.
   C. If both x and y are odd, then _________ is odd.
   D. If _________ is odd, then both x and y are odd.

Pedagogical context

The purpose of these problems is to help students learn about hypothetical propositions, how to determine their truth, and the diversity of possible antecedents and consequents. On the basis of students' previous learning, the teacher wants to help them clearly understand the definitions of *necessary condition, sufficient condition,* and *necessary and sufficient condition*.

Most school textbooks explain necessary condition and sufficient condition as follows:

When we have two conditions $p(x)$ and $q(x)$, and for all x,
$p(x) \Rightarrow q(x)$ holds,
we call $q(x)$ a necessary condition for $p(x)$ and $p(x)$ a sufficient condition for $q(x)$.

When both $p(x) \Rightarrow q(x)$ and $q(x) \Rightarrow p(x)$ hold or, in other words, when $p(x) \Leftrightarrow q(x)$ holds, we call $p(x)$ a necessary and sufficient condition for $q(x)$. In this case, we also say that the two conditions $p(x)$ and $q(x)$ are equivalent.

After giving such explanations, the textbooks provide exercises requiring students to judge which case is a necessary condition, a sufficient condition, or a necessary and sufficient condition for pairs of concretely given p (x) and *q (x)*.

Accordingly, students experience only the judging of the truth of given propositions; they are not involved in active situations where they have to figure out by themselves what is the antecedent or consequent of a proposition. Perhaps because of this practice, many teachers point out that despite repeated lessons on the definition of necessary condition and sufficient condition, students do not fully understand them. Full understanding may be obtained only through experiences in which students think through various cases and try to determine whether a proposition holds true for certain antecedents and consequents.

As a prerequisite to the topic of necessary condition and sufficient condition, this lesson was designed to help students learn about the diversity of possible antecedents and consequents for hypothetical propositions.

Expected Responses and Discussion of Them

Examples of expected responses

Problems 1 and 2 are used as a warm-up exercise for problem 3. Examples of expected responses are shown here only for the propositions in problem 3.

A. If one of x *or* y *is* odd and the other is even, then _________ is odd.

$x \pm y$

$(x \pm y)^n$, where n is a natural number not less than 2

$a(x \pm y)$, where a is odd

$x \pm y + k$, where k is even

$x^m \pm y^n + k$, where k is even

$ax \pm by$, where a and b are odd

$ax^n \pm by$, where a and b are odd

$xy + k$, where k is odd

B. If _________is odd, then one of x or y is odd and the other even.

$x \pm y$

$(x \pm y)^n$, where n is a natural number not less than 2

$a(x \pm y)$, where a is odd

$x \pm y + k$, where k is even

$x^m \pm y^n + k$, where k is even

$ax \pm by$, where a and b are odd

$ax^n \pm by$, where a and b are odd

C. **If both** x and y **are odd, then _________ is odd.**

xy

$x^m y^n$, where m and n are natural numbers

axy, where a is odd

$xy + k$, where k is even

$x \pm y + k$, where k is odd

$ax \pm by$, where a, b are odd and even, respectively

$ax^m \pm by^n$, where a, b are odd and even, respectively

$(x \pm y)^n + k$, where k is odd

$x^m \pm y^n + k$, where k is odd

D. If _________ is odd, then both x and y **are odd.**

xy

$x^m y^n$, where m and n are natural numbers

axy, where a is odd

$xy + k$, where k is even

Discussion of the responses

Problem 1 is a warm-up problem in which students judge whether a given proposition is true or not. Its purpose is to help students understand that propositions may be true or false according to differences in their antecedents or in their consequents, by proposing propositions that have the same antecedent but different consequents (propositions A, B, and C) and, conversely, those that have the same consequent but different antecedents (D, E, F). As for propositions G-J, in which the antecedents and consequents are interchanged, the purpose is for students to reconfirm what they have learned about the converse of a proposition.

In problem 2, students are led first to determine whether the sum, difference, or product of x and y fits as a consequent of a true proposition, and next to consider expressions of various combinations of x and y. It may serve as an introduction to problem 3.

In problem 3, students are led to consider suitable diverse expressions as an antecedent or a consequent. The fundamental expressions are the sum, the difference, and the product of x and y, and all their combinations are possible candidates. Therefore, although the expected responses are classified into types, strictly speaking there would not necessarily be a qualitative difference among types. When expressions proposed by the students have different forms, they should be regarded as different. It seems better to refrain from a hasty generalization or categorization.

What is considered as a further development

Judging whether a proposed expression is suitable as an antecedent or a consequent is not as easy as expected. This is true for expressions as an antecedent such as in propositions B or D in problem 3.

Substituting numerical values for variables may overcome these difficulties, Such difficulties also provide an opportunity to introduce the reduction to absurdity or the method of conversion. If students have already learned these methods of reasoning, problem 3 presents an opportunity to deepen their understanding by applying the methods.

Figure 6.35 summarizes the cases in problem 3 according to whether the proposition holds or not.

On the basis of this lesson, the teacher can develop the definitions of necessary condition, sufficient condition, necessary and sufficient condition, and equivalence in the next lesson.

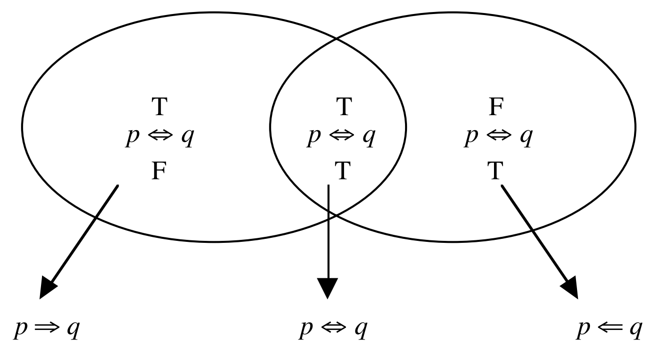

Fig. 6.35

Record of the Classroom Teaching

Teaching the lesson

The lesson was taught during one period before the topic "necessary condition and sufficient condition."

1. After distributing the worksheets, the teacher had students think about problem 1. (5 minutes)
2. The teacher asked students to present their own ideas and to confirm the correct answer. Then the teacher made sure that students understood the following points:
   - Even when statements have the same antecedents, they may be true or false according to the difference in their consequents
   - Even when statements have the same consequents, they may be true or false according to the difference in their antecedents.
   - Propositions G and H (and also propositions I and J) are mutually converse, and the converse of a proposition is not necessarily true if the original proposition is true. In addition to the converse, it is possible to formulate the inverse and contrapositive of a proposition. (5 minutes)
3. The teacher explained problem 2. The students considered various expressions by addition, subtraction, multiplication, and their combinations. (5 minutes)
4. The students presented the expressions that they had considered, and the teacher helped the class decide whether the expressions were true or false. (7 minutes)
5. The students considered problem 3 and submitted their worksheets after they had written down their

Problem 3

A	B	C	D
$x+y$	$3(x+y)$	xy	xy
$x-y$	$x+y^3$	$(xy)^2$	x^2y^2
$(x+y)^2$	$(x+y)(x-y)$	$x-y+1$	x^2y
$3(x+y)$	$3x+y$	$2x+y$	$3xy$
x^3-y^3	$(x+y)/2$ (false)	x^2+y^2-1	$x(y+2)$
$xy+1$		x^2+3x+y	$2x+y$ (false)
		$\lvert x^2-y^2\rvert$ (false)	

Table 6.3 summarizes, by type, the responses for problem 3 on forty-four students' worksheets.

Proposition	Type of Response	Number of Responses
A	$x \pm y$	44
	$(x \pm y)^n$, where $n \geq 2$	29
	$a(x \pm y)$, where a is odd	14
	$x \pm y \pm k$, where k is even	7
	$x^m \pm y^n + k$, where k is even	30
	$ax \pm by$, where a, b are odd	12
	$ax^m \pm by$, where a, b are odd	13
	$xy + k$, where k is odd	11
	Others	6
B	$x \pm y$	43
	$(x \pm y)^n$, where $n \geq 2$	17
	$a(x \pm y)$, where a is odd	11
	$x \pm y + k$, where k is even	7
	$x^m \pm y^n + k$, where k is even	22
	$ax \pm by$, where a, b are odd	8
	$ax^m \pm by$, where a, b are odd	12
	Others	3
C	xy	41
	$x^m y^n$, where m, n are natural numbers	29
	axy, where a is odd	12
	$xy + k$, where k is even	3
	$x \pm y + k$, where k is odd	12
	$ax \pm by$, where one of a, b is odd, the other even	18
	$ax^m + by$, where one of a, b is odd, the other even	4
	$(x \pm y)^n + k$, where k is odd	8
	$x^m \pm y^n + k$, where k is odd	3
	Others	6
D	xy	39
	$x^m y^n$, where m, n are natural numbers	23
	axy, where a is odd	9
	$xy + k$, where k is even	4
	Others	2

Table 6.3
Number of each Type of Response for Problem 3

Remarks after the lesson

High-achieving students made few responses. The reason may be that even when the students thought of many candidate expressions for an antecedent or consequent, they were puzzled as to whether the answers were suitable, perhaps because they could find no essential differences among the answers.

Furthermore, in problem 3, there were fewer responses and more wrong answers for propositions B and D than for propositions A and C. As expected, creating expressions to fit a consequent is more difficult than creating expressions to fit an antecedent. It is natural that fewer response types exist for given antecedents, so judging their truth is not easy. One approach to help students overcome these difficulties may be to teach the reduction to absurdity or the method of conversion. This lesson is very effective in motivating the students to learn such topics.

Students found great pleasure in these activities, which involve individually thinking of various expressions using their own free or natural ideas, presenting them to all the Students, discussing them with the other students, and formulating the new concepts called necessary condition and sufficient condition from the discourse. The open-ended teaching approach is an effective mathernatization process that can be used at the introductory stage of teaching concepts. By using this approach, students realize a way to mathernatize their freely proposed ideas, classify them, and gradually develop the ideas into mathematical principles or laws.

A **Similar Problem**

Integral Expressions

When $a, b, c,$ and d are consecutive integers in that order, $bc - ad = 2$ always holds. Following this example, write as many integral expressions in $a, b, c,$ and d as possible so that 0, 1, 2, or 3 appear to the right of the equals sign, as follows:

1. When the right side of the expression is 0
2. When the right side of the expression is 1
3. When the right side of the expression is 2
4. When the right side of the expression is 3