Investigation of Chebyshev Polynomials

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1 Introduction

Prior to taking part in this Mathematics Research Project, I have been responding to the Problems of the Week in the Math Forum Website. I am grateful that the Math Forum recommended me and gladly participated in this project. The course of the project was a challenge, as if I was in the dark searching for the correct path to take. The type of work I had to do was really somehow different from school work. Under the guidance of my mentor, Martin Kassabov of Yale University, and through his invaluable advice, I managed to tackle the numerous problems I came across. Before I begin to talk about the project itself, I would like to say that I really enjoyed working on my research. It has shed some light into the fascinating world of Mathematical research. There is no doubt that it has taken me a step further into the interesting World of Mathematics.

2 Mathematical Induction

Every statement must be justified by evidence. In math, this is done by proving that the equations obtained are correct. The first 'tool' I learned is Mathematical Induction. Induction is a step by step process of proving a statement; this method of constructing a proof is best suited to sequences or functions which are defined recursively. It is the process of generalizing from a repeated pattern from the past. What is perhaps a bit unusual about a proof by induction is that we must know the first result to construct the proof. Induction does not tell us how to construct a proof but gives us a way of proving it when it is known.

Induction works on expressions involving natural numbers, e.g.

$$1 + 3 + 5 + ... + (2n - 1) = n^2, \text{ for any natural number } n.$$

Let's call this statement $S(n)$, so

$$S(n) = 1 + 3 + 5 + ... + (2n - 1) = n^2, n \in \mathbb{N}$$

It could be shown that:

$$\begin{aligned} S(1) &= 1 &&= 1^2 \\ S(2) &= 1 + 3 &&= 2^2 \\ S(3) &= 1 + 3 + 5 &&= 3^2. \end{aligned}$$

We have now a formula which seems to work, but we must prove it.

A proof by induction really involves two steps.

1. $S(1)$ must be shown true.

2. Assuming that $S(n)$ is true, we need show that $S(n+1)$ must be true.

Once we show $S(1)$ holds, then we must prove that the statement $S(n)$ holds for any natural number n. So, if S(n) is true, we must show S(n + 1) is also true.

For instance,

$$\begin{array}{ll} \text{if } n=1, & \text{then } S(n+1) \text{ is true because } n+1=1+1=2; \\ \text{if } n=2, & \text{then } S(n+1) \text{ is true because } n+1=2+1=3; \\ \text{if } n=3, & \text{then } S(n+1) \text{ is true because } n+1=3+1=4. \end{array}$$

Following this logic, any natural number would eventually be reached.

The example above required us only to prove the first term, S(1), to be true, to assume S(n) to be true, and to prove S(n + 1) to be true. Sometimes it is easier to show that $S(n+1)$ is true if we assume both $S(n-1)$ and $S(n)$. If we want to make mathematical induction like this, we need to show that both $S(1)$ and $S(2)$ are true in order to start the process. This is the case if a function is defined recursively by 2 previous functions. The polynomial central to this project is one such example. Let's define these Chebyshev Polynomials and call them T_n. $T_n(x) = 1$ for n = 0, $T_n(x) = x$ for n = 1, and $T_n(x) = 2x(T_{n-1})(x) - T_{n-2}(x)$ for $n > 1$. T_n will be used throughout the discussion that follows and refers to the Chebyshev Polynomials as defined.

3 Some properties of the polynomials T_n

Some $T_n(x)$-es are listed below:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 + 5x \end{aligned}$$

From these formulas we can see some interesting patterns in $T_n(x)$ and the graph of $T_n(x)$.

1. The exponents of x with non-zero coefficients in T_n are alternatively odd and even and the degree of the polynomials is equal to n.

   This could be shown by induction:

   Step 1: The statement is clearly true for $n = 0$ and $n = 1$.

Step 2: Let us show that it also holds for $n+1$, assuming that it is true for n and $n-1$. Let us also assume that n is even, the case when n is odd is similar.

The polynomial $T_{n-1}(x)$ has all exponents of x as odd, by assumption. The polynomial $T_n(x)$ has all exponents of x as even, also by assumption. Then

$$T_{n+1}(x) = 2x \cdot T_{n-1}(x) - T_n(x) \text{ by definition for } n > 1,$$

$$T_{n+1}(x) = 2x(T_n)(x) - T_{n-1}(x) = 2x(\text{even power}) - \text{odd power}$$
$$= \text{some odd power} - \text{odd power} = \text{odd power}.$$

2. The graph of T_n passes through the point (1,1) for any n, i.e. $T_n(1) = 1$.

$T_0(x) = 1$	by definition for $n = 0$
$T_1(x) = x$	by definition for $n = 1$
$T_n(x) = 2x(T_{n-1})(x) - T_{n-2}(x)$	by definition for $n > 1$

Step 1:	$T_0(1) = 1$	by definition
	$T_1(1) = 1$	by definition
Step 2:	$T_{n-1}(1) = 1$	by assumption for $n-1$
	$T_n(1) = 1$	by assumption for n
	$T_{n+1}(1) =$	assuming that $x = 1$
	$= 2(1)(T_n)(1) - T_{n-1}(1)$	
	$= 2(1)(1) - 1 = 1$	

This proves that $T_n(1) = 1$ holds for every n, i.e, the graphs of the polynomials $T_n(x)$ pass though the point $(1, 1)$.

3. $T_n(x)$ is an even function, when n is even, also $T_n(x)$ is an odd function, when n is odd.

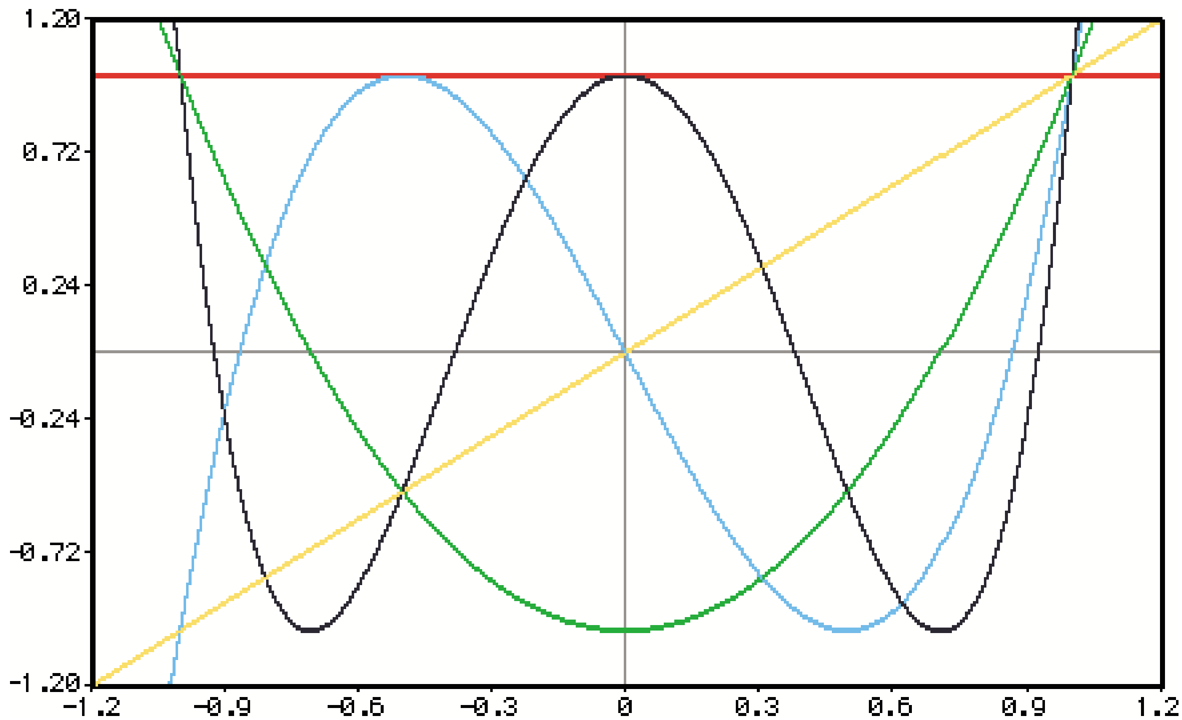
4. Notice that the exponent of the highest degree increases by 1 as n increases by 1, this means that the number of stationary points (critical points or turning points) will increase by 1. As the graph of $T_n(x)$ for all n are positive, and from the fact that there are different number of turning points, the graph will tend either to the second or third quadrant.

   This also implies that $T_n(-1)$ will have a different value for a different n. Using Induction, it can be proved that $T_n(-1)$ is either $+1$ or -1 for all n.

5. For a graph with the polynomial of highest degree n, the maximum number of stationary points is $n-1$. Drawing the graphs of $T_n(x)$ will reveal that all the $n-1$ turning points occur within the domain $\{-1 < x < 1\}$ and have a value of either -1 or 1. This fact reveals that the most important region of $T_n(x)$ is the interval $[-1, 1]$.

6. There is also a pattern in the number of waves that go up or down. For a polynomial of highest degree n, if n is an even number, there is one more 'up wave' than 'down wave'. If n is an odd number, there is an equal number of them.

7. There are some graphs with common turning points. It turns out that the turning points of $T_n(x)$ are also some of the turning points of $T_{kn}(x)$. Note: kn means a multiple of n, e.g. the critical points of $T_2(x)$ are also in $T_4(x)$ as well as $T_8(x)$ etc.



Graph of some T_n

T_0 (red), T_1 (yellow), T_2(green), T_3 (light blue), T_4 (dark blue)
As indicated above, the graph of $T_n(x)$ for every n passes through the point $(1, 1)$.

4 Turning points

Literally, it is the turning points under investigation. Figuratively, it is also the turning point of the project.

Finding a generalization in the turning points may make it easier to determine other points in T_n. I began studying the largest $x_n < 1$ for which $T_n(x_n) = 1$.

A list of n with its corresponding x is as follows:

n	x_n
2	-1
3	-0.5
4	0
5	0.308
6	0.500
7	0.623
8	0.707
9	0.769
10	0.808
11	0.841
12	0.866

The values were determined from the graph. Since they are defined recursively i.e., $T_4(x)$ is defined from $T_3(x)$ and $T_2(x)$, therefore finding a relationship between x and x_n will facilitate in analyzing the polynomials. The table of values was studied closely and the appearance of some sines and cosines of simple angles like:

$$\sin 30^\circ = 0.5, \quad \sin 45^\circ = 0.707, \quad \text{and} \quad \sin 60^\circ = 0.866$$

gave an impetus to use trigonometric functions to describe the relation.

Using trial and error, it was found that the for each n the number x_n, such that x_n is the largest less than 1 and the graph of $T_n(x)$ passes though point $(x_n, 1)$, is given by

$$x_n = \cos(360^\circ/n).$$

This discovery is the key to the project, since it suggested that there is a closer relationship between polynomials T_n and trigonometric functions. At this point I had no idea why this relation held and how I could prove it.

As Martin suggested that to convince myself that the equation is correct, I tried to find equations for similar relationships. And I succeeded in finding similar formula for similar sequences x_n defined using the graphs of $T_n(x)$.

1. For the largest x_n such that $T_n(x_n) = 0$, i.e., the point $(x, 0)$ lies on the graph of T_n. For each n we have that $x_n = \cos(90^\circ/n)$.

2. For the largest x_n such that $T_n(x_n) = -1$, i.e., the point $(x, -1)$ lies on the graph of T_n. For each n we have that $x_n = \cos(180°/n)$.

3. For the second largest x_n such that $T_n(x_n) = 1$, i.e., the point $(x, 1)$ lies on the graph of T_n. For each n we have that $x_n = \cos(720°/n)$.

Martin noticed the following interesting fact.

$$\begin{aligned}
T_n(\cos(0°/n)) &= \quad 1 \quad = \cos(0°) \\
T_n(\cos(90°/n)) &= \quad 0 \quad = \cos(90°) \\
T_n(\cos(180°/n)) &= \ -1 \quad = \cos(180°) \\
T_n(\cos(360°/n)) &= \quad 1 \quad = \cos(360°) \\
T_n(\cos(720°/n)) &= \quad 1 \quad = \cos(720°) \\
T_n(\cos(360K°/n)) &= \quad 1 \quad = \cos(360K°)
\end{aligned}$$

This fact suggested that there is a relationship like $T_n(\cos(x/n)) = \cos x$. The proof for the equation $T_n(\cos(x/n)) = \cos x$ is the critical part of the project.

5 Proof of the relation $T_n(\cos x) = \cos(nx)$

After guessing a nice formula for the polynomials T_n, we need to prove that it holds.

Prove: $T_n(\cos x) = \cos(nx)$. (Note that this is equivalent to saying that $T_n(\cos(x/n)) = \cos x$.)

My proof is based on the identity

$$\cos(A + B) = 2\cos A \cos B - \cos(A - B).$$

The proof of $T_n(\cos x) = \cos(nx)$ holds for every n is done by induction:

Step 1:	$T_0(\cos x) = 1 = \cos(0x)$	by definition
	$T_1(\cos x) = x = \cos(1x)$	by definition
Step 2:	$T_{n-1}(\cos x) = \cos((n-1)x)$	by assumption for $n - 1$
	$T_n(\cos x) = \cos(nx)$	by assumption for n
	$T_{n+1}(\cos x) = \cos((n+1)x)$	equation to be proven

In order to prove the above equation we need to expand both sides and show that they are equal to one and the same thing.

The right side is equal to:

$\cos((n+1)x) = \cos(nx + x) =$	distributive law
$= 2\cos(nx)\cos x - \cos(nx - x) =$	using the trig identity
$= 2\cos(nx)\cos x - \cos((n-1)x)$	distributive law,

and the left side is:

$T_{n+1}(\cos x) =$	substituting T_n and T_{n-1}
$2\cos x\cos(nx) - \cos((n-1)x)$	in the formula for T_{n+1}

Therefore the left and the right sides are equal, which proves that

$$T_{n+1}(\cos x) = \cos((n+1)x),$$

assuming that $T_{n-1}(\cos x) = \cos((n-1)x)$ and $T_n(\cos x) = \cos(nx)$.

Thus we have shown by induction that

$$T_n(\cos x) = \cos(nx) \text{ holds for every } n \in \mathbb{N}.$$

This proof uncovered an alternative way to compute $T_n(x)$ for x in the interval $[-1, 1]$.

These polynomials are known as Chebyshev Polynomials.