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INVESTIGATIONS of CHEBYSHEV POLYNOMIALS

INTRODUCTION

Mathematics is used to explore and discover the way things work. It is the basis for many scientific fields. Arnold Ross, a professor of mathematics at Ohio State said, "Science floats on a sea of Mathematics" (Ross, 1998). The main goal of this research is to find patterns in polynomials and relate them to other fields of science and mathematics. In particular, what are Chebyshev polynomials, what properties do they hold, and what characteristics do they carry when graphed and analyzed.

Polynomials are defined as a function P(x) that can be written in the form :

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$

Oved Shisha, a prominent mathematician in the second half of the twentieth century, said, "Polynomials are temperamental creatures. If you force them to behave somewhere, they will go wild in other places." (Shisha, 1977) This quote explains how polynomials have many different characteristics and can be changed and manipulated giving you many different outcomes. A group of polynomials in mathematics are known as Chebyshev polynomials. They are polynomials defined as $T_n(x)$ such that cos (nθ) = $T_n$cos(θ)(Dash, 2001). These polynomials were named after Pafnuty Chebyshev who first studied them and did made significant breakthroughs with them. Pafnuty Chebyshev was a Russian mathematician (1821-1894) who founded the St. Petersburg mathematical school and also studied the theory of probabilities, quadratic forms, orthogonal functions, the theory of integrals, gearings, the construction of geographic maps, and formulas for the computation of volumes (Mac tutor, 2000).

Chebyshev polynomials are like fine jewels that reveal characteristics under illumination from varying positions. They have many properties that make them useful in these areas of mathematical analysis: interpolation theory, orthogonal polynomials, approximation theory, numerical integration, numerical analysis, ergodic theory, and number theory (Rivlin, 1974). They also are very useful to computer musicians and people in the field of kinematics. Chebyshev polynomials form one class of orthogonal polynomials. Orthogonal polynomials are defined as: any of an infinite collection of special polynomials $P_n(x)$, with the subscript n indicating the degree of the polynomial. Their importance derives from there being solutions of differential equations and from the possibility of representing any

(continuous) function as a (possibly infinite) sum of these polynomials, a device often useful in the differential equations used in physics and engineering (Britannica, 2001). Other types of orthogonal polynomials are: Hermite polynomials, Legendre polynomials, and Laguerre polynomials.

Chebyshev polynomials start off as trigonometric functions in the form $\cos(n\theta)$ and then are transformed into polynomials using the relationship:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

This is because you can manipulate these trigonometric functions into polynomials by simplifying with the formula stated above and substitute x where $x = \cos(\theta)$. Trigonometry is defined as the branch of mathematics concerned with specific functions of angles and their application to calculations in geometry. The subject of trigonometry was developed from a need to compute angles and distances in such fields as astronomy, map making, surveying, and artillery range finding. The 6 basic trigonometric functions are: sine, cosine, tangent, cotangent, cosecant, and secant.

Kim-Chuan Toh and Lloyd N. Trefethen researched Chebyshev polynomials of a matrix. The study of these polynomials was motivated by the analysis of Krylov subspace iterations in numerical linear algebra. An algorithm was presented for computing these polynomials based on reduction to a semi definite program that would then solve them by a primal-dual interior point method (SIAM Journal on Matrix Analysis and Applications, 1998). Also, M. Alvo and P. Cabilio researched the calculation of hyper geometric probabilities using Chebyshev polynomials (Cabilio and Alvo, 1999). Alain Fournier and John Buchanan did research in the field of computer graphics and computer-aided design. They used Chebyshev polynomials for boxing and intersections of parametric curves and surfaces (Imager, 1994). The extensive research done with Chebyshev polynomials has led to many breakthroughs and helped bring new ideas and topics to think about. Through mathematical ideas many scientific breakthroughs occur, and as Pafnuty Chebyshev once said, "To isolate mathematics from the practical demands of the sciences is to invite the sterility of a cow shut away from the bulls."

DERIVATION OF NON-RECURSIVE FORM OF CHEBYSHEV POLYNOMIALS

Chebyshev polynomials are defined as: $T(n)=\cos(nt)$, for $n > 0$ and t just a variable

To begin with, we use the fact that $\cos(a+b)= \cos(a)\cos(b)-\sin(a)\sin(b)$

From this it follows that:

$$\cos(nt + t) = \cos(nt)\cos(t) - \sin(nt)\sin(t)$$

$$\cos(nt - t) = \cos(nt)\cos(t) + \sin(nt)\sin(t)$$

By adding these equations together and you get the new identity:

$$\cos(nt + t) + \cos(nt - t) = 2\cos(nt)\cos(t)$$

using the distributive property you can re-write it as:

$$\cos((n+1)t) + \cos((n-1)t) = 2\cos(nt)\cos(t)$$

So:

$$\cos((n+1)t) = 2\cos(nt)\cos(t) - \cos((n-1)t)$$

By substitution of $x=\cos(t)$ and the definition of chebyshev polynomials:

$$T(n+1, x) = 2xT(n, x) - T(n-1, x)$$

or a recursive formula for the Chebyshev polynomials.

Next, to develop a non-recursive formula:

We know:

$T(0,x) = 1$
$T(1,x) = x$
$T(n,x) = 2xT(n-1, x) - T(n-2, x)$ for integers $n > 2$ by the recursive formula
Because it's a geometric series, let us assume $T(n,x) = t^n$

Then by the recursive formula and substitution :

$$t^n = 2xt^{(n-1)} - t^{(n-2)}$$

Dividing through the equation by $t^{(n-2)}$ gives you

$T^2 = -1 + 2t^{(2-n)} x t^{(n-1)}$

Or

$T^2 - 2xt + 1 = 0$

By the quadratic formula you get the solutions:

$$t = x + \sqrt{x^2 - 1} \qquad t' = x - \sqrt{x^2 - 1}$$

Because of the way we derived it, the sequence t^n and the sequence $(t')^n$ will both satisfy the recurrence relation of the Chebyshev Sequence, and so will any constant multiple of those two sequences => the sum of constant multiples of the two sequences will satisfy the two sequences.

In other words, $A[t^n] + B[(t')^n]$ will satisfy the recurrence relation for any constants A and B.

or

$T(n,x) = A[t^n] + B[(t')^n]$

The last step is completing the formula and find A and B so that this expression "gives the right answer" when $n = 0$ and $n = 1$. That is, we must solve the simultaneous linear equations:

$1 = At^0 + BT'^0 = A + B$

$x = At^1 + Bt'^1 = A(x + \sqrt{x^2 - 1}) + B(x - \sqrt{x^2 - 1})$

... Algebra ...

$A = 1/2$ and $B = 1/2$

THEREFORE :

$T(n, x) = A[t^n] + B[t'^n]$ or replaced with A, B, t, and t' which we derived above

$T(n,x) = [(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]/2 = [t^n + t'^n]/2$

as a closed form formula for the Chebyshev Polynomials in terms of x.

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