Probabilistic number theory : A report on work done…

What is the probability that a randomly chosen integer has no square factors?

We can construct an initial formula to give us this value as follows:
If a number is to have no square factors, then it must not be a multiple of the square of any prime.

P(integer is multiple of a square prime, p^2) = $\frac{1}{p^2}$

…(if integers are to be chosen randomly).

Therefore,
P(number is not a multiple of a square prime, p^2) =

$$1 - \frac{1}{p^2}$$

In applying this to all integers, we must find the probability, P, that a number is not a multiple of p_1^2 or p_2^2 or p_3^2 or … or p_n^2.

For example: The probability that a number is not divisible by 4, 9, or 25 (ie. the squares of the first three primes), is

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right) = \frac{16}{25}$$

A simple numerical calculation reveals that the product of the first 10 terms is 0.61234…

This approach takes account of numbers which have the square of a composite number, as a factor. This composite will, in accordance with the fundamental theorem of arithmetic (see page 3), be evaluable as the product of primes, and thus such a number would be included in this method.

Clearly,

$$P = \prod_{For\ all\ primes\ p,\ from\ 2\ to\ \infty} \left(1 - \frac{1}{p^2}\right)$$

Each term in this product can be evaluated as a geometric series, summed to infinity.

Derivation of the formula for the summation of an infinite series

Consider the evaluation of the product:

$$Q = (1 - x)(1 + x + x^2 + x^3 + x^4)$$
$$= (1 + x + x^2 + x^3 + x^4) - x(1 + x + x^2 + x^3 + x^4)$$

From visualizing how these brackets will expand, it is clear that:

$$Q = 1 - x^5$$

Now in fact:

$$(1 - x)(1 + x + x^2 + x^3 + x^4 + \ldots + x^n) = 1 - x^{(n+1)}$$

Dividing by $(1 - x)$:

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

So suppose it was replaced by r where:
-1 <r<1.
Then, in the limit, As $r^{n+1} \to 0$, and so:

$$\frac{1 - r^{n+1}}{1 - r} \to \frac{1}{1 - r}$$

Therefore, as $n \to \infty$

$$\sum_{r=0}^{\infty} r^k = \frac{1}{1 - r}$$

when -1 <r<1

Consider the formula for the sum to infinity of the infinite series:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

Now let r = $\frac{1}{p^2}$ (clearly -1<r<1)
Substituting this into the formula gives:

$$\sum_{k=0}^{\infty} \left(\frac{1}{p^2}\right)^k = \frac{1}{1 - \frac{1}{p^2}}$$

So it seems the original component of the product P appears as the denominator of the RHS. Therefore:

$$\frac{1}{P} = \prod_{pprime} \frac{1}{1 - \frac{1}{p^2}} = \prod_{pprime} \sum_{k=0}^{\infty} \left(\frac{1}{p^2}\right)^k$$

Consider the expansion of $\frac{1}{P}$ as products:

$$\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\ldots\right)\bullet$$
$$\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots\right)\bullet$$
$$\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\ldots\right)\bullet$$
$$\left(1+\frac{1}{7^2}+\frac{1}{7^4}+\frac{1}{7^6}+\ldots\right)\bullet$$
$$\vdots$$

Clearly the computation of this product is going to involve the summing of many products, each of which will contain a term from each of the series bracketed above. One such product will for example be:

$$\frac{1}{2^6\bullet 1\bullet 5^2\bullet 7^8}$$

…and in general each denominator will be the product of squares of primes.

Consider the Fundamental Theorem of Arithmetic, which states that every integer n can be expressed as the product of primes, in a unique way, such that:

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_k^{e_k}$$

And therefore:

$$n^2 = p_1^{2e_1} p_2^{2e_2} p_3^{2e_3} \ldots p_k^{2e_k}$$

…as in the denominators.

Due to the fact that each representation of n^2 is unique, this means that each unique denominator can be expressed as n^2.

Therefore:

$$\boxed{\frac{1}{P}=\sum_{n=1}^{\infty}\frac{1}{n^2}}$$

Clearly this series is not geometric, as the ratio between the terms is not constant. Therefore another method is necessary to evaluate the sum of the reciprocals of the squares.

The search for this sum is not new. Since the time of Euler, a number of methods have been suggested, one of which is given here. It turns out, perhaps rather surprisingly, that the sum:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

The Evaluation of: $\frac{1}{p} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Giesy, D. *Mathematics Magazine* 1972

This sum is easily calculable to a finite degree of accuracy through basic arithmetic, but for centuries mathematicians have been finding methods of evaluating the exact sum. Here is one such method.

Trigonometry, as indicated by the appearance of π, is a major component.

Consider the following trigonometric sum A:

$$\sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + 2\sin\frac{x}{2}\cos 3x + \ldots + 2\sin\frac{x}{2}\cos nx$$

Using the identity:

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

Combined with the fact that :

$$\sin(-x) = -\sin x$$

The sum A can be re-written as:

$$\sin\frac{x}{2} + (\sin\frac{3x}{2} - \sin\frac{x}{2}) + (\sin\frac{5x}{2} - \sin\frac{3x}{2}) + (\sin\frac{7x}{2} - \sin\frac{5x}{2}) + \ldots + (\sin(n+\frac{1}{2})x) - (\sin(n-\frac{1}{2})x)$$

Clearly this sum will telescope to:

$$\sin(n+\frac{1}{2})x$$

Therefore:

$$\sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + 2\sin\frac{x}{2}\cos 3x + \ldots + 2\sin\frac{x}{2}\cos nx = \sin(n+\frac{1}{2})x$$

Dividing both sides by

$$2\sin\frac{x}{2}$$

Gives:

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \ldots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

Consider the result of multiplying both sides by x and integrating with respect to x:

$$\int_0^{\pi}\frac{x}{2}\,dx + \int_0^{\pi} x\cos x\,dx + \int_0^{\pi} x\cos 2x\,dx + \int_0^{\pi} x\cos 3x\,dx + \ldots + \int_0^{\pi} x\cos nx\,dx = \int_0^{\pi}\frac{x\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

Evaluating the LHS:

$$\int_0^{\pi}\frac{x}{2}\,dx = \frac{\pi^2}{4}$$

Each of the next integrals can be evaluated through integration by parts:

$$\begin{aligned}\int_0^{\pi} x\cos mx\,dx &= \left[\frac{x\sin mx}{m}\right]_0^{\pi} - \frac{1}{m}\int_0^{\pi}\sin mx\,dx \\ &= \left[\frac{1}{m^2}\cos mx\right]_0^{\pi} \\ &= \frac{1}{m^2}((-1)^m - 1) \\ &= \begin{cases} -\frac{2}{m^2} \text{ if } m \text{ is odd} \\ 0 \text{ if } m \text{ is even} \end{cases}\end{aligned}$$

Therefore, the sum of the integrals on the previous page will not change for even values of n.
Hence we may assume n is odd, say $n = 2t + 1$
Making this substitution gives:

$$\frac{\pi^2}{4} - 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots + \frac{1}{(2t+1)^2}\right) = \int_0^{\pi} \frac{x\sin(2t+\frac{3}{2})x}{2\sin\frac{x}{2}}\,dx$$

As $t \to \infty$ the LHS is equal to $\frac{\pi^2}{4}$ minus twice the sum to infinity of the reciprocals of the odd squares.
The RHS will tend to zero, given the following theorem of Riemann.

Riemann's Lemma
For any function, f, which is continuous, and differentiable on $(0, \pi)$, the integrals:

$$\int_0^{\pi} f(x)\sin tx\,dx \quad \text{and} \quad \int_0^{\pi} f(x)\cos tx\,dx$$

both go to 0 as $t \to \infty$

The integral, $\int_0^{\pi} \frac{x\sin(2t+\frac{3}{2})}{2\sin\frac{x}{2}}\,dx$ may be written as: $\int_0^{\pi} \frac{\frac{x}{2}}{\sin\frac{x}{2}}\sin(2t+\frac{3}{2})x\,dx$

Since $\dfrac{\frac{x}{2}}{\sin\frac{x}{2}}$ is continuous and differentiable on (0, π), this can be transformed into the sum of two integrals like those in Riemann's lemma by applying the addition formula for sine

to $\sin(2t+\frac{3}{2})x$

Therefore:

$$\frac{\pi^2}{4} - 2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...\right)=0$$

Or:

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...$$

The 'missing terms', i.e. the sum of the reciprocals of the even powers are a multiple of the entire sum of the reciprocals of squares:

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+... = \frac{1}{4}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...\right)$$

Also:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+... = \frac{\pi^2}{8}$$

Therefore, adding the two equations above we have:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+... = \frac{\pi^2}{8}+\frac{1}{4}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...\right)$$

Or in sigma notation:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

That is:

$$\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$$

Therefore:

$$\frac{1}{p} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Therefore:

$$P = \frac{6}{\pi^2} = \textit{The probability that an integer has no square factors}$$

Taking pie to 4 sf, this gives the value of P as 0.6077… not far from the 0.61234 we obtained from the earlier numerical simulation.

Conclusion

This was by far the most hefty piece of mathematics I had ever attempted! Working through this problem involved a mixture of skills. My own mathematics manipulation played an important part in the first section of my work. The whole concept of probabilistic number theory was new to me, and the guidance of my mentor was important in homing my efforts, and helping my out when I was stuck.

The summation of 1/ n^2 involved a change in what my work involved. I began to read the mathematics that was sent to me over email, and work through it myself, trying to understand it. The dialogue with my mentor focussed on unravelling the problems in what I had read, as new concepts were explained to me.

Overall, work on this project led me to discover and explore an area of mathematics that was entirely new to me, and introduced me to a range of new ideas.

I would be happy to discuss anything written here. Drop me a line at:

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