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Pattern Trains

Mathematical Focus

- Input-output tables
- Number patterns
- Variables in mathematical expressions

Students draw a variety of “pattern trains” on triangular grid paper and calculate the perimeter of the train for different lengths. They look for patterns in the growth of the perimeter and make predictions about the perimeter for a pattern train of length \( L \). After repeating this process for six types of pattern trains, students make a graph of Perimeter vs. Length for all six trains. They describe the rate of increase of the perimeter in each table as \( L \) increases by 1, and compare this to the slope of the graph. (If they don’t already know how to compute the slope of a line, they will learn it in this activity.)

Triangle Pattern Train

Preparation and Materials

- Student Page 1: Pattern Trains
- Student Page 2: Triangle Grid Paper
- Graph paper
Part 1: Finding the Perimeter of a Triangle Train

1. Make triangle trains and find the perimeter of different length trains.

Ask students to look at the triangle at the top left of Student Page 1: Pattern Trains. Explain that the activity involves drawing pattern trains, using different shapes, and finding their perimeters. Ask:

- What is the perimeter of the triangle? [3]
- If you fill in the lines of the second triangle to make a connected triangle train, what is the perimeter of the train? [4—interior lines don’t count as part of a perimeter]

![Triangle Train Diagram]

2. Record the perimeter of different length triangle trains in a chart.

Create a chart like the one below. Fill in the data for a 1-train and a 2-train. Have students continue the process by adding more triangles to the train. Ask:

- What is the perimeter now? Add the length and the perimeter of this train to the chart. [Length is 3, perimeter is 4]
- Can you predict the perimeter of the next train? Draw one more triangle and check your prediction, recording your results on the chart. [Length is 4, perimeter is 5]
3. Look for patterns in the data and describe a formula for finding the perimeter of a triangle train given its length.

Ask students the following questions:

- Do you see a pattern in how the perimeter increases as the train increases by one triangle? Students should observe that the perimeter increases by 1 every time the length increases by 1.

- What do you think the perimeter will be for a 5-triangle train? A 6-triangle train? Can you predict the perimeter of a 10 triangle train? A 50-triangle train? If students do not recognize that the perimeter is two more than the length, ask them to compare each length in the table with the corresponding perimeter.

- Can you tell me a rule or formula so that if you know the length of a train you can tell what the perimeter is and be sure you are right? How can you be sure you are right?

Students may offer a variety of acceptable responses to explain their rule or formula for finding the perimeter of a train given its length. The rule is something like: The perimeter is 2 more than the length. In explaining, students might look back at the drawing and say: Each
train has a slanted line at each end (that makes 2), and each new shape adds one more horizontal line, so the total perimeter is always the number of triangles plus 2. Or, they might say: Every time a new triangle is added, it adds two lines and removes one (the previous end, which is now an interior line).

4. Write the formula for finding the perimeter of a triangle train as a mathematical rule.

Ask students to write their formula as a mathematical rule using P as the perimeter and L as the length. Students should be able to write \( P = L + 2 \). If they cannot do this, ask them to write out the relationship in words, and then to rewrite it using letters to stand for the perimeter and length and mathematical symbols to show equality and the operation of addition.

5. Given the perimeter of a triangle train, find its’ length.

Present students with the following challenges:

- If a triangle train has a perimeter of 12, what is its length? [10]
- If a train has a perimeter of 50, what is its length? [48]
- Could you ever have a triangle train with a perimeter of 1? Of 2? [No, in both cases, because the shortest possible train, with just one triangle, has a perimeter of 3]

Part 2: Pattern Trains from Different Shapes

1. Explore the relationship between train length and train perimeter for trains made from different shapes.

Give students a copy of Student Page 1. Ask them to go through the process outlined in Part 1 for each of the trains on the worksheet: square, rhombus, trapezoid, and hexagon. For each case have students describe the pattern and write a formula for the perimeter of a train of any length. They should then be able to tell you the perimeter for any length, and the length for any perimeter, of each particular train.

\[
\begin{align*}
L &= 1 & 2 & 3 \\
P &= 4 & 6 & 8 \\
P &= 2L + 2
\end{align*}
\]
Square train: Start: Perimeter of one square is 4. Pattern: perimeter increases by 2 every time length increases by 1. Formula: \( P = 2L + 2 \).

\[
\begin{array}{ccc}
L & 1 & 2 & 3 \\
P & 4 & 6 & 8 \\
\end{array}
\]

Rhombus train: Start: Perimeter of one rhombus is 4. Pattern: Perimeter increases by 2 every time length increases by 1. Formula: \( P = 2L + 2 \).

\[
\begin{array}{ccc}
L & 1 & 2 & 3 \\
P & 5 & 8 & 11 \\
\end{array}
\]

Trapezoid train: Start: Perimeter of one trapezoid is 5. Pattern: Perimeter increases by 3 every time length increases by 1. Formula: \( P = 3L + 2 \).

\[
\begin{array}{ccc}
L & 1 & 2 & 3 \\
P & 6 & 10 & 14 \\
\end{array}
\]
Hexagon train: Start: Perimeter of one hexagon is 6. Pattern: Perimeter increases by 4 every time length increases by 1. Formula: \( P = 4L + 2 \).

2. Explore characteristics that are common to all five patterns and rules.

If students have been able to find and explain the patterns and rules for all five trains, ask them to think about these questions:

- **What’s the same about all five patterns and rules?**
  Most students will notice that all have +2 added as part of their formulas. Some students will notice that in all four cases the perimeters increase by a constant amount. This is a characteristic of a linear relationship or a linear function, which is true of all the examples they will study in this unit. A few students may notice that the number that \( L \) is multiplied by in each formula is the same as the amount the perimeter increases from one train to the next. This is also a characteristic of linear relationships.

- **How does the amount of increase (or the number multiplied by \( L \)) relate to the perimeter of the starting polygon?** The following chart may help students answer this question:

<table>
<thead>
<tr>
<th>Starting Shape</th>
<th>Starting Perimeter</th>
<th>Constant Increase in Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Square</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Rhombus</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Students may notice that the amount of constant increase is equal to 2 less than the starting perimeter. Ask students to predict what the pattern and rule would be for a train made of octagons (or another shape with perimeter 8).
Make pattern trains by combining two or more different shapes and building a train with the combined shape.
Crossing the River

Mathematical Focus

- Linear relationships
- Generalizations from examples
- Patterns in problem-solving

Students explore patterns involved in a series of classic mathematical problems called Crossing the River. Two children and eight adults must cross a river in a small rowboat. The first problem involves finding the number of times the boat must be rowed back and forth across the river in order to get everyone to the other side. The next problem involves finding a formula to give the number of trips needed for any number of adults and just 2 children. The final problem challenges students to find a formula for the number of trips needed for any number of adults and children to cross the river.

Preparation and Materials

- Student Page 3: Crossing the River
- Graph paper
- A paper cut-out boat and different-colored buttons or other markers to represent adults and children (optional)

It would be helpful if students have already completed Activity 1 or have some familiarity with representing situations using words and variables.
**Part 1: Eight Adults and Two Children**

1. Introduce “Crossing the River” and begin to discuss strategies for solving the problem.

Give students a copy of Student Page 3: Crossing the River. Read the problem and look at the picture. Ask: *How can you figure out how many crossings it would take to get everyone across the river?*

Students may need to spend some time thinking about how to get everyone across. By using a question-and-answer process, you can help them understand that a child must always be the one to row back to get the next adult, because two adults cannot fit in the boat.

If students seem to be confused at first about how to approach the problem, you can make the problem concrete by drawing the river on a piece of paper, using different-colored buttons, cubes, or pebbles—eight of one color to represent the adults, and two of another color to represent the children. Cut out a small boat of paper or cardboard, and have students actually ferry the “people” across and back as they complete the table suggested in the next step.

2. Create a table for recording information about the trips made back and forth across the river.

Ask students to make a table showing the number and direction of crossings, who is in the boat, the number of adults and children who have already crossed over, and the number remaining to cross.

<table>
<thead>
<tr>
<th>Crossing/Direction</th>
<th># in Boat</th>
<th># Crossed over</th>
<th># Remaining</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \rightarrow )</td>
<td>( A + C )</td>
<td>( 1A + 1C )</td>
<td>( 7A + 1C )</td>
</tr>
</tbody>
</table>
3. Look for patterns in the data.

As students complete the chart, they may begin to see that on every odd-numbered crossing, the number of adults crossed over increases by 1. They should eventually figure out that for 8 adults (and 1 child) it will take 15 crossings, plus 2 more crossings to get the last child, for a total of 17.

### Part 2: Solving the Problem for Any Number of Adults and Children

1. Generalize the Crossing the River problem for any number of adults and two children.

Ask: *Can you figure out how many crossings it would take for any number of adults and 2 children?* This will require making a new table that show the number of trips for different numbers of adults and just 2 children. They can start with the data collected in Part 1 to get the data for the number of crossings for different numbers of adults, and then add 2 crossings for the second child.

<table>
<thead>
<tr>
<th>Number of Adults</th>
<th>Number of Crossings (for two children)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 + 2 = 3</td>
</tr>
<tr>
<td>2</td>
<td>3 + 2 = 5</td>
</tr>
<tr>
<td>3</td>
<td>5 + 2 = 7</td>
</tr>
<tr>
<td>. . .</td>
<td></td>
</tr>
</tbody>
</table>
2. Predict the number of crossings needed for different numbers of adults.

Once students have filled in the chart for as many as 8 adults and 2 children, ask: *Can you predict how many crossings would be needed for the following numbers of adults, all with two children: 10 adults? [19 + 2 = 21] 20 adults? [39 + 2 = 41] 100 adults? [199 + 2 = 201]* They should write their predictions on the table.

Challenge students to describe the relationship between numbers of adults and numbers of crossings as a rule or formula. They could also represent the relationship as a function machine with the number of adults as the input and the number of crossings as the output. The resulting relationship is \( N = 2A + 1 \), where \( A \) is the number of adults and 2 is the constant number of children.

3. Explore the number of crossings needed for various numbers of children.

Ask students to think about how they could find the number of crossings needed for any number of children:

- *How many crossings would it take with \( A \) adults and 0 children?* [The problem has no solution because two adults cannot fit in the boat. There must be at least one child in order for more than one adult to get across.] With \( A \) adults and 1 child, they should be able to see from their existing data that it takes \( 2A - 1 \) crossings. With \( A \) adults and 2 children, it takes \( 2A + 1 \).

- *How many more trips will it take for each additional child?* [2 more trips will be needed for each additional child] The following table may help them think about the problem:
<table>
<thead>
<tr>
<th>Number of Children (C)</th>
<th>Number of Crossings with A Adults</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Impossible</td>
</tr>
<tr>
<td>1</td>
<td>2A – 1</td>
</tr>
<tr>
<td>2</td>
<td>2A + 1</td>
</tr>
<tr>
<td>3</td>
<td>2A + 3</td>
</tr>
<tr>
<td>4</td>
<td>2A + 5</td>
</tr>
<tr>
<td>8</td>
<td>2A + ??</td>
</tr>
<tr>
<td>10</td>
<td>2A + ??</td>
</tr>
<tr>
<td>C</td>
<td>2A + ??</td>
</tr>
</tbody>
</table>

4. Look for patterns in the data.

Once students have worked out the number of crossings for A adults and 3 and 4 children, ask: *By how much does the number of crossings increase for every additional child?* [2] Ask: Can you use the number of crossings needed for 4 children to predict the number needed for A adults and 8 or 10 children? [2A + 5 + 2 x 4 = 2A + 13, for eight children; 2A + 17, for 10 children]

5. Describe the relationship between the number of children and the number of crossings.

Ask students to describe in words the relationship between the number of children and the number of crossings? [Twice the number of adults plus twice the number of children, then subtract three] Have them build a function machine or write an equation for this relationship. [\(x + 2A - 3\), or \(N = 2A + 2C - 3\)]

Challenge students to think back to the description of the original situation and explain why the linear relationship has the form it does. Ask them to think again about how many trips it takes for A adults and just 1 child (2A – 1). [For each additional child, 2 more are added. Since 1 child is included in the first relationship, 2(C – 1) additional trips are needed for C children. This makes the total number of trips \(N = 2A - 1 + 2(C - 1) = 2A + 2C - 3\)]
**Function Machines and Mystery Machines—1**

**Overview**

**Mathematical Focus**
- Functions as input-output rules
- Concrete representations of functions
- Linear functions

Students explore the use of function machines to represent sequences of computations that can be used repeatedly with many different numbers. A function machine is an imaginary device with an input hopper in which a number can be placed, and an output spout through which a result comes out of the machine. The figures below show a \([\times 3]\) machine, which multiplies each input by three and a combined function machine that multiplies each input by 3, adds 2, and then outputs the result.
Preparation and Materials

Before the session, gather the following materials:

- Student Page 4: Function Machines—Simple Machines
- Student Page 5: Function Machines—Combined Machines
- Student Page 6: Function Machines—Combined Machines
- Graph paper
Part 1: Exploring Function Machines

1. Create and investigate simple function machines.

Explain to students that this activity involves exploring function machines. Have them read Student Page 4: Function Machines—Simple Machines, and then draw several different simple function machines (that is, machines that perform just one operation: addition, subtraction, multiplication, division, squaring etc.), recording the results of several different inputs in an input-output table. After trying a few machines that perform whole number operations, encourage students to construct machines that perform fraction or decimal operations, such as $\times (1/3)$, $-3.75$, $\times 0.25$, or $\div 0.1$, or machines that include negative numbers, such as $\times (-2)$ or $\div (-5)$.

Encourage use of negative numbers, fractions, and decimals as inputs to the machines. Working with fractions, decimals, or negative numbers in function machines gives students an opportunity to review and practice fraction and decimal arithmetic.

2. Explore combined machines.

Read Student Page 5: Combined Machines with students. Have them make up several combined machines and describe how they work. Here are some interesting questions to explore with combined machines:

- What happens if you reverse the order of the machines in a combined machine? Do the results change? Are there some combined machines for which the results do not change? Which ones?
If a machine involves both addition or subtraction and multiplication or division, the order of the machines changes the input-output table when the machines are rearranged. For example, a \([\times 3 + 2]\) is not the same as a \([+ 2 \times 3]\) machine. If a machine involves only combinations of additions and subtraction, or only combinations of multiplication and division, changing the order does not make a difference. For example, a \([\times 3 \div 2]\) gives the same results as a \([\div 2 \times 3]\) machine.

- **Can you make two equivalent machines—machines that give the same output for any input?** In addition to reversing multiplication/division or addition/subtraction machines, there are many other ways to make equivalent machines. For example, \([\times 2 + 4]\) is equivalent to \([+ 2 \times 2]\).
- **Can you make a combined machine for which the output is the same as the input?** Can you make a machine that does this, using two simple machines? Three? Four? These are machines that use inverses. \([+ 3 - 3]\) is an example; \([\times 3 \div 3]\) is another; \([\times (-1) \times (-1)]\) is a third.

**3. Create a function machine for pattern trains.**

Have students look at the formulas for perimeters and lengths of pattern trains they made in Activity 1. Ask: *Can you make a function machine that gives you the perimeter for a particular train as output, using the length of a train as an input?* For example, for a square pattern train the formula was \(P = 2L + 2\); this is the same as a \([\times 2 + 2]\) machine. If you use any positive integer as input, the output will be the perimeter of a square train with that number of shapes.

**Part 2: Mystery Machines**

1. Solve a mystery machine together.
Ask students to read Student Page 6: Function Machines—Mystery Machines. Work through the last example on the page together. When students think they know the rule connecting the input and the output, ask them to construct a machine that will give the same inputs and outputs as the mystery machine.

2. **Make up and solve mystery machines.**

Play this as a two-person game; one person is the machine-maker and one is the guesser. The machine-maker creates a combined function machine made up of two or more machines. The guesser then gives an input, and the machine-maker must calculate the output. When the guesser thinks he or she knows the rule or formula relating the input and the output, the guesser should build a machine to carry out that rule and test that it gives the same output as the mystery machine for any input. When this happens, the mystery rule is known and the players switch roles. It’s important to switch roles, because there is a lot to learn from being the machine-maker as well as the guesser. The machine-maker may use a calculator or do paper and pencil calculations.

The mystery machines activity provides opportunity to develop and discuss strategies. Typical student strategies start with random guessing; next, students use a regular sequence of inputs (1, 2, 3, 4, 5 or 10, 20, 30, 40); next, they learn to always start with zero to get the constant term. Eventually, students may realize that the output changes by a constant amount as the input increases by 1, and that same constant amount is the multiplier of the variable in the algebraic representation of the mystery machine.

If the student already knows about linear equations of the form $y = mx + b$, this can be a big help for solving mystery machines that involve multiplication and addition or subtraction. The constant term, $b$, can be found by using an input of zero. The coefficient of $x$ can be found by choosing two inputs that differ by 1 and seeing how much the output changes. The change in output for a difference of 1 in the input is the same as $m$, the coefficient of $x$ in the equation.

**Extension**

Play mystery machines with a spreadsheet. Set up the spreadsheet with three columns. The first column is for the input. The second
column has a mystery formula that simulates a mystery machine. The third column is left blank until the guesser thinks he or she knows the rule and enters a formula in that column. If the outputs for the guesser’s rule are the same as the mystery outputs, the guess is correct.
Function Machines and Mystery Machines—2

Overview

Mathematical Focus

- Function machines and linear functions
- Graphs of linear functions
- Slopes of linear functions
- Patterns

In this activity, students make and guess mystery machines, as in the previous activity. This time, in addition to creating a machine to carry out the mystery machine’s work, students plot a graph of input-output pairs on a Cartesian grid. They also write mathematical rules to describe their machines. For example, a \([x \times 3 + 2]\) machine can also be described by the rule \(y = 3x + 2\), where \(x\) is the input and \(y\) is the output.

This activity helps students develop connections between linear equations and their graphs. Students come to understand that for all linear equations, the graph of solutions forms a straight line; that the slope of the line is equal to the amount the output changes as the input increases by 1; and that the “y-intercept” of the line is the constant term in the equation (the output when the input is zero).

Preparation and Materials

*Before the session, gather the following materials:*

- Student Pages 4, 5, and 6, as a reference
- Paper for recording results in tables and for drawing compound machines
- Graph paper with axes in the center with the horizontal and vertical axes labeled “x” and “y,” respectively
- A ruler
- A calculator

Notes

Students must have had prior experience with function machines, compound machines, and mystery machines (see Activity 3). Students should also have prior experience plotting points on a coordinate grid.
Part 1: Graphing the Input and Output of Function Machines

1. Play mystery machines with graphs and equations.

Create a simple mystery machine such as \([\times 2]\). As students choose inputs, ask them to record inputs and outputs in a table as before. This time, write “x” above the input column and “y” above the output column. For every input-output pair recorded, plot the number pair as a point on a graph. The input value is measured along the horizontal (x) axis, and the output value along the vertical (y) axis.

2. Look for patterns in the table.

After a few input-output combinations, ask: *Can you see a pattern in the table?* [One possible pattern is that the output increases by 2 as the input increases by 1; another is that the output is two times the input.] *Can you use the pattern to predict what the output will be for the next input you try?*

3. Look for patterns in the graph.

Ask: *Can you see a pattern in the graph?* [For example, the points lie on a straight line] If they cannot see a pattern, ask them to choose a few more inputs. If students see that the points lie along a line, ask them to use a ruler to connect the line and extend it beyond the existing points.

4. Use the graph to predict more input output pairs.

Once the line is drawn, ask students to identify the x and y coordinates of another point on the line—one that has not already been plotted (for example, [2,1]). Ask: *What do you think the output of the mystery machine will be if you choose 2 as the input?*

It is important that the lines be drawn carefully so that the graphs can be used to predict other input-output pairs.
5. Find the slope of the line on a graph.

Show students how to find the slope of the line on the graph by locating two points; finding the vertical distance (rise) between the points and the horizontal distance (run) between the points; and dividing the rise by the run to determine the slope. Ask: Can you see a connection between the slope and the machine you made for the mystery machine? [The slope of the line should be 2, the same as the multiplier in the [× 2] machine.]

6. Construct a compound function machine.

Now choose a compound mystery machine that starts with the same multiplication machine, for example [× 2 – 1]. Work with students on the input-output pairs until they can construct and draw a compound function machine that creates the same input-output pairs as the mystery machine. Ask students to find the slope of the graph and compare it to the machine they constructed. [The slope of the line should still be 2, the same as the multiplier in the [× 2 – 1] machine.]

7. Make several compound machines and plot input-output pairs.

Make several more compound machines starting with the same [× 2] machine: [× 2 + 5], [× 2 – 3], and so forth. For each machine, plot enough input-output points on a table and plot coordinates on the same graph. The result should be a series of lines parallel to one another. Ask students to find the slope of each line. [All slopes are the same, 2]. Ask: Can you see a pattern in the lines on the graph and the function machines? [The slope of the line is the same as the number the input is multiplied by.]
8. Look for relationships between the y-intercept and the function machines.

Now ask students to identify the points where each line crosses the y-axis. Ask: *Can you see a connection between the y value of one of those points and the function machine that produced that particular line?*” [The value of y when x = 0 is called the y-intercept. It is also the output of a machine when the input is zero. It is also the same as the number added or subtracted to the multiplier in the compound mystery machine.]

**Part 2: Using Graphs to Solve Mystery Machines**

1. Use a graph to solve a mystery machine.

Choose a mystery machine that begins with a multiplication or division machine, followed by an addition or subtraction machine. (Examples: $\div 2 - 3$, $\times 4 + 1$, $\times 1 - 5$)

For each mystery machine, make an input-output chart and a graph. As soon as there are enough points on the graph, ask students to draw a line, calculate the slope, and record the y-value where the line crosses the y-axis.

Students should then be able to identify a mystery machine with the slope providing the multiplier for the first machine, and the y-intercept providing the value of the addition or subtraction machine.

Students should check for accuracy by trying their machine to verify that it gives the same outputs as the mystery machine.

2. Solve mystery machines with different types of multipliers.

Once students have the knack of this process, try some mystery machines with negative multipliers. (Examples: $\div (-2) + 1$, $\times (-1) + 4$, $\times (-2) - 5$)
Finally, try some mystery machines that do not start with a multiplier. (Examples: \([+ 4 \times (-2)], [- 1 \div 2 + 3]\). If students follow the same procedure, they will construct a compound machine that has the same effect as your machine but uses different components. Introduce the concept of equivalent machines—two machines that always produce the same outputs for the same inputs. (Examples: \([+ 4 \times (-2)] = [\times (-2) - 8]; [- 1 \div 2 + 3] = [\div 2 + 2.5]\))

**Part 3: Using Equations to Represent Mystery Machines**

Now that you and your students are familiar and comfortable with function machines, you can learn to represent them as equations. Begin with a verbal description of what the machine does, and translate that to a mathematical statement. Remind students that \(x\) represents the input and \(y\) represents the output. Point out that some of the equations representing function machines have been simplified. Tell students that simplifying an equation is a way of creating two or more equations that are equivalent to one another. Two equations—like two function machines—are equivalent if all the input-output pairs are identical for both equations.
<table>
<thead>
<tr>
<th>Function Machine</th>
<th>Description in Words</th>
<th>Mathematical Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>([\times 2])</td>
<td>To get the output, multiply the input by 2</td>
<td>(Y = 2x)</td>
</tr>
<tr>
<td>([\times 2 + 3])</td>
<td>To get the output, multiply the input by 2 and add 3 to the result</td>
<td>(Y = 2x + 3)</td>
</tr>
<tr>
<td>([+ 3 \times 2])</td>
<td>To get the output, add 3 to the input and multiply the result by 2</td>
<td>(Y = 2(x + 3) = 2x + 6)</td>
</tr>
<tr>
<td>([-3 ÷ 3 + 3])</td>
<td>To get the output, subtract 3 from the input, divide the result by 3 and add 3 to the final result</td>
<td>(Y = \frac{(x - 3)}{3} + 3 = x ÷ 3 + 2)</td>
</tr>
<tr>
<td>([\times (-2) + 6 ÷ 2 - 3])</td>
<td>To get the output, multiply the input by (-2); add 6 to the result; divide that result by 2; and subtract 3 from the final result</td>
<td>(Y = \frac{(-2x + 6)}{2} - 3 = -x + 3 - 3 = -x)</td>
</tr>
</tbody>
</table>

Now that students can connect machines with equations and graphs, it's time to connect *expressions* with their graphs. Here are the same examples with information about the graphs added to the chart:
<table>
<thead>
<tr>
<th>Function Machine</th>
<th>Description in Words</th>
<th>Mathematical Equation</th>
<th>Graph: Slope and Y-Intercept</th>
</tr>
</thead>
</table>
| \([\times 2]\)    | To get the output, multiply the input by 2 | \(y = 2x\) | slope: 2  
y-intercept: 0 |
| \([\times 2 + 3]\) | To get the output, multiply the input by 2 and add 3 to the result | \(y = 2x + 3\) | slope: 2  
y-intercept: 3 |
| [+ 3 \times 2]   | To get the output, add 3 to the input and multiply the result by 2 | \(y = 2(x + 3) = 2x + 6\) | slope: 2  
y-intercept: 6 |
| \([-3 \div 3 + 3]\) | To get the output, subtract 3 from the input, divide the result by 3, and add 3 to the final result | \(y = (x –3) \div 3 + 3 = x \div 3 + 2\) | slope: 1/3  
y-intercept: 2 |
| \([\times (-2) + 6 \div 2 - 3]\) | To get the output, multiply the input by \(-2\); add 6 to the result; divide that result by 2; and subtract 3 from the final result | \(y = (-2x + 6) \div 2 - 3 = -x + 3 - 3 = -x\) | slope: -1  
y-intercept: 0 |

3. Guess a mystery machine with the smallest number of guesses.

The easiest way is to think of a linear function as an equation in the form \(y = mx + b\), where \(m\) is the slope, or rate of change of the function, and \(b\) is the y-intercept, or constant term. Then, using 0 as an input, students will find the constant term in the equation immediately. If they use 1 as the second input, the difference between the two outputs (for \(x = 0\) and \(x = 1\)) will provide the rate of change, that is, the value of \(m\). Therefore students should—after some practice—be able to identify a mystery machine using only two inputs, 0 and 1. If students can do this—and understand why it works—they will have achieved mastery of linear functions.
Now students are familiar with five different ways of describing any linear function:

- Function machines
- Input-output table
- Descriptions in words
- Graphs
- Equations

Give students information about a linear function in one form and ask them to construct one or more of the other representations. For example:

- Give them a table and ask them to build a function machine
- Give them a graph and ask them to write an equation
- Give them a function machine and ask them to draw a graph
- Give them a written description and ask them to make a table
Number Tricks

Mathematical Focus

- Inverse operations
- Simplification of mathematical expressions
- Descriptions, function machines, and mathematical expressions as representations of sequences of operations

Number tricks are special function machines that do one of two things: 1) always produce an output number equal to the input number, or 2) always give the same output, no matter what input number you start with. Students explore several number tricks and describe how they work, first in words, and finally as function machines, then as mathematical equations.

Preparation and Materials

*Before the session, gather the following materials:*

- Student Page 7: Number Trick Cards, made up on individual index cards
- A master copy of Student Page 7 (for use by mentor)
- Paper for recording results and drawing function machines

Notes

Students should have some experience with function machines (Activity 3) before beginning this activity.
Part 1: The “Pick a Number” Trick

1. Test out a number trick to see what happens to the number you pick.

Begin by reading Card 1 from Student Page 7: Number Trick Cards. Ask students to pick a number but not to reveal each separate mental calculation until they have the final result. When students have completed the sequence once, ask them to pick another number. Continue until students are convinced that the result is always the number they started with.

If students have difficulty with the mental arithmetic, suggest writing down each step and its result on a piece of scrap paper.

2. Try to explain how the number trick works.

Ask: Can you explain why the trick works? (Most likely they will not be able to explain it because it is difficult to recall all the steps after concentrating on the mental arithmetic.) Now let students read Card 1, and ask again if they can explain why the trick works. Some students will be able to explain at this point that when they add 3, then multiply by 2, and then subtract 6, they have nothing left but the original number. They may also be able to explain that when they multiply the number by 2, and later divide by the same number, they get exactly the number they started with. (It is not important for students to be able to make a full explanation at this point.)

3. Draw a function machine that represents the number trick.

[The result should be [+ 3 × 2 – 6 ÷ 2].] Ask them to pick a different number and then trace through the workings of the machine, step by step. Ask again if they can explain why the trick works.

4. Write a mathematical equation that represents the number trick.
Suggest using \( x \) to stand for the number someone picks. It’s important to stress that each new operation is applied to the total of all the previous operations. It’s easier to write the expressions if you make a table like this:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pick a number</td>
<td>( x )</td>
<td></td>
</tr>
<tr>
<td>Add 3</td>
<td>( x + 3 )</td>
<td>( 2x + 6 )</td>
</tr>
<tr>
<td>Multiply by 2</td>
<td>( 2(x + 3) )</td>
<td>( 2x + 6 )</td>
</tr>
<tr>
<td>Subtract 6</td>
<td>( 2(x + 3) - 6 )</td>
<td>( 2x )</td>
</tr>
<tr>
<td>Divide by 2</td>
<td>( \frac{2(x + 3) - 6}{2} )</td>
<td>( x )</td>
</tr>
</tbody>
</table>

Notice that when you simplify the expression you get \( x \).

5. Examine some different number tricks.

Repeat steps 1–4 using Cards 2 and 3 and those that you invent yourself. As students work through the examples, help them understand that \( x \) will always be the result for this type of number trick—regardless of the number of steps. Help students understand why this is always true and how the trick works. [It works because you are always “undoing” an operation by using the inverse operation (e.g., + undoes −, × undoes ÷, etc.).]

Part 2: The Second Type of “Pick a Number” Trick

1. Determine what happens with a different type of number trick.

Read Card 4. Ask students to pick a number but not to reveal each separate mental calculation until they have the final result. When students have completed the sequence once, ask them to pick another number. Continue until students are convinced that the result is always 7, no matter what number they started with.

If students have difficulty with the mental arithmetic, suggest writing down each step and its result on a piece of scrap paper.

2. Explain how the second kind of number trick works.
Ask: *Why do you always get an output of 7?* Let students read Card 4, and ask again if they can explain why the trick works. Some students will be able to explain at this point that when they multiply by 3, add 6, and then take 1/3 of the result, they will have their original number + 2. Then, if they subtract their original number and add 5, they will always have 7 remaining, no matter what number they started with. (It is not important for students to be able to make a full explanation at this point.)

3. **Write a mathematical expression the second type of number trick.**

Suggest using x to stand for the number someone picks, the input. It’s important to stress that each new operation is applied to the total of all the previous operations. It’s easier to write the expressions if you make a table like this:

<table>
<thead>
<tr>
<th>Pick a number</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiply by 3</td>
<td>3x</td>
</tr>
<tr>
<td>Add 6</td>
<td>3x + 6</td>
</tr>
<tr>
<td>Take 1/3 (i.e., divide by 3)</td>
<td>(3x + 6)(1/3) = x + 2</td>
</tr>
<tr>
<td>Subtract the original number</td>
<td>x + 2 - x = 2</td>
</tr>
<tr>
<td>Add 5</td>
<td>2 + 5 = 7</td>
</tr>
</tbody>
</table>

Notice that when you simplify the equation you get 7.

4. **Explore some more number tricks.**

Repeat the process, using Cards 5 and 6 and those that you invent yourself. As students work through the examples, help them understand that a constant will always be the result of the equation for this type of number trick—regardless of the number of steps. Help students understand why this is always true and how the trick works. [It works because you are always subtracting your original number (x) and combining this with operations that multiply and divide by the]
same number to make the process more difficult to follow. That is, you are using inverse operations to “undo” the previous steps.]

The second type of “Pick a Number” Trick, those for which the result is a constant, cannot be easily represented by function machines.
Invent Your Own “Pick a Number” Trick

Now that students have had some experience with Pick a Number tricks, they may be able to create some of their own. Remind them of a few things to help them do this:

- If you add something, always subtract it later in a different form (and vice-versa).
- If you multiply by something, always divide by it later (and vice versa).
- Don’t make the mental computations too difficult for your audience. Use numbers that will make the mental computations relatively easy—for example, if you are going to divide by 3, add or subtract 6 or 9 or some other multiple of 3 so that people won’t have to do arithmetic with fractions in their heads.
- You may prefer to work out the equation form before the written description. In that way, you can be sure the result will always be \( x \) (for the first type of trick) or a constant (for the second type of trick).
- Once you know more difficult to follow—for example if you multiply by 2, take \( \frac{1}{2} \) of the result later; triple the result instead of multiplying by 3; take \( \frac{1}{4} \) of a result instead of dividing by 4; and so forth.
- Try the trick with a friend and make sure it works before launching it on the general public.
Modeling Realistic Situations

Mathematical Focus

- Mathematical rules, graphs, and input-output tables as ways to describe linear relationships
- Variables in realistic situations
- Slopes and vertical intercepts for linear relationships

Students explore realistic situations that involve linear relationships, such as the total cost at a certain price per item, the amount of work done at a constant rate, or the number of items left at a constant rate of depletion. For each situation, students start with whatever information is given and work from there to ultimately describe and write a mathematical rule, make a graph of that rule, make an input-output chart for the rule, and answer questions about the situation.

Preparation and Materials

Before the session, gather the following materials:

- Student Page 8: Realistic Situations
- Paper for recording results in tables and graphs

Notes

Students should have prior experience with function machines, compound machines, and mystery machines. They should also have experience working with graphs and creating graphs.
Part 1: Mowing Lawns—Starting with Input-Output Chart

1. Examine the relationship between time spent mowing and the number of lawns mowed.

Refer to Student Page 8: Realistic Situations. Ask students to look at the chart in Part 1. Tell students that they will use the chart to think about the relationship between the time Matt spent mowing and the number of lawns he could mow. Point out that Matt mowed at a constant speed, and then ask the students to think about how the inputs and outputs are related. As they answer questions about the number of lawns mowed in different amounts of time, have them fill in the new pairs on the input-output chart. If students are having trouble conceptualizing what the chart means, remind them that it is similar to a mystery machine. Students should look for the function that would turn the input (time) into the output (number of lawns). Ask:

- What mystery function would relate the two columns?
- How many lawns could Matt mow in 4 hours? [8 lawns]
- How much time would it take Matt to completely mow 12 lawns? [6 hours or 360 min.]
- If Matt worked for 8 hours during the day, could he mow 17 lawns during a single day? [no]
- Can you write a mathematical rule to describe the relationship between time spent and lawns mowed? [Student will need to choose letters to represent each of the variables in the formula. One example is \( L = \frac{1}{30} \times T \) or \( T = 30L \), where \( L \) stands for lawns mowed and \( T \) stands for time spent mowing in minutes.]

2. Graph the relationship between time and lawns mowed.

Ask students to take a piece of graph paper and graph the series of points they calculated for the relationship between time spent and lawns mowed. Ask:
• What do you think should go on the horizontal axis of this graph, and what should go on the vertical axis? [Ask students to explain the answer they give. This graph could go either way; the number of lawns could depend on the amount of time spent, or the amount of time spent could depend on the number of lawns mowed.]

• What numbers will you place along the time axis and the number of lawns axis? [The unit sizes of the two axes will be very different given the large numbers of minutes for the time and the small numbers for the lawns.]

• Plot the points that you have in your input-output chart on the graph. What happens if you connect the points?

• Predict what number will correspond with 1.5 hours on the graph. Now look at your graph. Does that number of lawns correspond with 1.5 hours?

• Use your graph to fill in some more pairs in your input-output chart. Do these new pairs fit with the relationship you determined?

3. Examine a more advanced lawn mowing relationship.

Here is a more advanced question that students can try to answer:

• What if Matt had a friend to help him mow the lawns and his friend mowed lawns at the same rate as Matt? What would this mean in terms of how many lawns they could mow together every hour? [The time to mow a lawn would be cut in half, so twice as many lawns could be mowed in the same amount of time. The revised formula is \( T = 15L \) or \( L = (1/15)T \)

You may also wish to use the questions found in the Extension section that refer to the relationship between, and the purpose of, graphs, input-output charts, and mathematical rules.

Part 2: Candy Bars—Starting with a Graph

1. Examine and explain a graph that shows the results of a candy bar sale.
Have students examine the graph on Part 2 of Student Page 8, which shows the number of candy bars left to be sold as days of the sale go by. Ask them to explain what the different parts of the graph mean. Ask:

- Where on the graph do you look to find out how many candy bars there initially were to sell? [Vertical intercept, 400 candy bars]
- How many candy bars were left to sell after 12 days? [0 candy bars]
- Were candy bars sold at a constant rate during this sale? How can you tell? [Yes, there is a straight line on the graph to represent the relationship]
- Is the slope of this graph positive or negative? What is the slope of this graph? [The slope is negative because the number of candy bars decreases as the number of days increases. The slope is \(-\frac{100}{3}\), which is the change in the vertical axis over the change in the horizontal axis.]

2. Make an input-output chart to go with the candy sale graph.

If you think your students can handle more of a challenge, have them describe the relationship between the two variables and write a mathematical rule straight from the graph. Otherwise, ask:

- Can you make an input-output chart that will convey the same information the graph conveys? [Remind students that the input is represented by the horizontal axis and the output is represented by the vertical axis.]
- What are some points from the graph that you could put in the input-output chart and that would be important for displaying information? [The vertical intercept, the horizontal intercept]
- How many candy bars are left after 3 days? [300]
- How many days have gone by when there are 100 candy bars left? [9]

3. Explain the relationship between days and candy bars left.
Have students describe, in words, the relationship between the number of days that have passed and the number of candy bars left.

Ask students to write a mathematical rule for the they just described, which is also shown on the graph and the input-output chart. [In order to make a mathematical rule, the students will need to choose letters to stand for each variable. Students may choose to draw a function machine in order to help make this rule. One example is \( C = 400 - \frac{100}{3}D \), where \( C \) is the number of candy bars and \( D \) is the number of days.]

4. Answer more advanced questions about the candy bar sale.

Here is a more advanced question, if you think your students are ready:

- How would the graph and mathematical rule change if you started with 100 candy bars instead of 400 but sold the candy bars at the same rate as when starting with 400? [The vertical intercept would be at 100. The rule would be \( C = 100 - \frac{100}{3}D \). You would sell candy bars for 3 days before running out.

In this problem of decreasing candy bars with time, students may need help in recognizing that as the input increases, the output decreases, which means that a negative coefficient is required for the input in the mathematical rule.

You may wish to use the questions found in the Extension section that refer to the relationship between, and the purpose of, graphs, input-output charts, and mathematical rules.

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Part 3  Buying Bagels—Starting with a Mathematical Rule

1. Examine a bagel buying scenario.

Read aloud the situation in Part 3 of Student Page 8. Ask:

- How much does the total cost of the order increase for each additional bagel purchased? [50 cents]
How does the price of the cream cheese affect the total cost of the order? [$2.50 is added to whatever amount needs to be paid for the bagels]

If you start by choosing a letter to represent each variable, can you write a mathematical rule to describe this situation? [Students may prefer to start by making a function machine, for example, \( T = (0.50)B + 2.50 \), where \( T \) is the total cost, \( B \) is the number of bagels, and $2.50 is the price of the package of cream cheese]

2. **Generate an input-output chart to represent the bagel buying situation.**

Ask:

- What is the input column and what is the output column of this chart? [Input is number of bagels, output is total cost]
- How much total money is needed if you are buying 5 bagels? [$5]
- How about if you are buying 3 bagels? [$4]
- You have $12 to spend, and you would like to spend it all when you buy bagels and cream cheese. How many bagels should you order? [19]

3. **Make a graph to represent the bagel buying linear relationship.**

Ask:

- What would be on the horizontal axis and what would be on the vertical axis of this graph? [Horizontal axis is number of bagels, vertical axis is total cost of order]
- What would the vertical intercept be? Why? [Make sure that students are thinking about the cost of the cream cheese in answering this question. $2.50 is the vertical intercept.]
- Would the slope of this graph be positive or negative? [The slope should be positive because increasing numbers of bagels means increasing cost. If students don't understand the slope concept, then talk about what would happen to the total cost if more bagels were added to the order.]
Have the students plot some points, connect them, and see if the graph matches their answers to the previous questions. If not, ask students to re-examine their answers and think the problem through again.

4. Answer some more advanced questions.

What would happen if the cream cheese was bought by the ounce rather than by the container? The price is $2.50 per ounce of cream cheese, and 2 ounces of cream cheese are needed for every bagel. Bagels are still 50 cents apiece. How would you rethink the mathematical rule for the situation? [This relationship involves two variables, so $T = (\$0.50)B + (\$2.50)C$, where $T$ is the total cost, $B$ is the number of bagels, and $C$ is the number of ounces of cream cheese. If the student gets this far, the next step is to realize that $C$ equals $2B$ because 2 ounces of cream cheese are needed for every bagel.]

You may wish to use the questions found in the Extension section that refer to the relationship between, and the purpose of graphs, input-output charts, and mathematical rules.

1. Give students more practice at moving between graphs, charts, and mathematical rules that represent realistic situations by giving them more problems that are linear relationships. These relationships could include speed of walking or driving (distance vs. time), pencils sharpened (# of math problems completed vs. # of times pencil had to be sharpened while working on them), sales tax (cost of sales tax vs. cost of item), or anything else you choose. For example, the relationship between pencils sharpened and math problems completed could be stated to the student as an equation, $S = (1/15)P + 1$ where $S$ is the number of times the pencil is sharpened if the pencil must be sharpened once at the beginning and then after every 15 problems, and $P$ is the number of problems completed.

2. Ask students to try to come up with their own realistic situations involving linear relationships. Remind them that they could start by graphing the situation, by making a mathematical rule for the relationship, or by making an input-output chart like they did for mystery functions. You could choose to give students a mathematical rule and ask them to formulate the realistic situation based on that.  

rule. For example, you could say that \( y = 10x - 25 \), and ask them to come up with a situation that could be represented by this equation. Alternatively, you could say that the rule is \( C = 10B - 25 \), where \( C \) is the cost of your order, \( B \) is the number of books being ordered, and 25 is the amount of the gift certificate you have at the book store. The students could then formulate a story that is represented by this equation, given the context you have provided.

3. Have students think about the different ways that they have now represented realistic situations involving linear relationships (through input-output charts, graphs, and mathematical rules). Ask:

- Do input-output charts, graphs, and mathematical rules represent the same relationships?
- In what situations would you want to use an input-output chart, as opposed to a graph or a mathematical rule, to represent a linear relationship, and vice versa?
- If you start with any one of the three representations, can you get all the information you could get from either of the other representations?
Golden Apples

Mathematical Focus

- Doing-Undoing
- Development of general formulas from specific cases

Students explore the patterns involved in another classic mathematical problem, called Golden Apples. Students first work backward to solve this problem, then they attempt to solve the problem for any number of apples the prince brought home.

Preparation and Materials

Before the session, gather the following materials:

- Student Page 9: Golden Apples
- Paper for recording results in tables and graphs
Part 1: Two Ways to Solve the Problem

Have the students read Student Page 9: Golden Apples. Give the students plenty of time to work on Question 1. (Listed below are two ways to solve the problem.)

The first way that you can have the students think about the problem is to have them solve the problem by working backwards. This means figuring out how many apples the prince had before he met the third troll, then the second, then the first.

Here are some ways you can help students think about this (if they need help):

- You need to solve three smaller problems, each of which looks the same but uses different numbers. If the prince has $B$ apples before he meets the troll, each troll takes $B \div 2 + 2$ apples from the prince.
- If the prince has $A$ apples left after meeting the troll, then $A = B \div 2 - 2$.
- You can find $B$ if you know $A$ by rewriting the equation, adding 2 to both sides, and multiplying both sides by 2, to get $B = 2(A + 2)$.
- Now make a chart showing the number of apples the prince had before and after meeting each troll. After meeting the last troll, 2 apples were left, so start with this number and fill out the whole chart:

Make a table like this:
Check the result by starting from your answer, 44, which should be the number of apples before meeting any of the trolls:

44: $\frac{1}{2} \times 44 - 2 = 22 - 2 = 20$

20: $\frac{1}{2} \times 20 - 2 = 10 - 2 = 8$

8: $\frac{1}{2} \times 8 - 2 = 4 - 2 = 2$

In Question 2, students must write an equation to represent the solution to the problem. That is, they start with the original number of apples picked and make an equation for the number of apples the prince had with him when he got home. It will help to make a table. This time, let $P$ be the number of apples the prince picked from the tree.

<table>
<thead>
<tr>
<th>Troll</th>
<th># of Apples Before Meeting Troll</th>
<th># of Apples Left After Meeting Troll</th>
</tr>
</thead>
<tbody>
<tr>
<td>Last troll</td>
<td>P</td>
<td>$\frac{P}{2} - 2$</td>
</tr>
<tr>
<td>Middle troll</td>
<td>$\frac{P}{2} - 2$</td>
<td>$(\frac{1}{2})(\frac{P}{2} - 2) - 2$</td>
</tr>
<tr>
<td>First troll</td>
<td>$\frac{P}{4} - 3$</td>
<td>$(\frac{1}{2})(\frac{P}{4} - 3) - 2$</td>
</tr>
</tbody>
</table>

Now all that’s left to do is solve the equation $\frac{P}{8} - \frac{7}{2} = 2$ (since the prince had 2 apples left). Multiply both sides of the equation by 8, to get $P - 28 = 16$; then add 28 to both sides, to get $P = 44$. 
Students can use this formula to answer the second question.
Suppose the prince came home with N apples. How many did he pick?
Just set \( P = \frac{8}{2} - 7/2 = N \), then \( P - 28 = 8N \), and \( P = 8N + 28 \).

This equation can be used to answer the third question. If the prince came home with whole apples only (no fractions of apples), what are the possible numbers of apples the prince picked? This can also be answered using a table:

<table>
<thead>
<tr>
<th>Number of Apples the Prince Came Home with (N)</th>
<th>Number of Apples the Prince Picked (P = 8N + 28)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>28</td>
</tr>
<tr>
<td>1</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>44</td>
</tr>
<tr>
<td>3</td>
<td>52</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
</tr>
<tr>
<td>\ldots</td>
<td></td>
</tr>
</tbody>
</table>

If the prince picked any number of apples other than those that satisfy the formula, he would have come home with fractions of apples, according to the payment demanded by each troll.
Pattern Trains

Triangle trains

Square trains

Rectangle trains

Hexagon trains
Triangle Grid Paper
Crossing the River

Two children and eight adults are on a hike, when they come to a wide river they must cross to get to their destination. A small rowboat has been left near the trail. The boat can hold two children, or one adult, or one adult and one child. It is not big enough for two adults. Fortunately, both children are good rowers, as are all eight adults.

1. How many times must the hikers row the boat across the river in order to get everyone across? (A round trip counts as two crossings.)

2. Find a mathematical rule that will tell you how many crossings are needed for two children and any number of adults.

3. Find a mathematical rule that will tell you how many crossings are needed for any number of children and any number of adults.
Function Machines—Simple Machines

A function machine is an imaginary mathematical machine, a little bit like a very special computer or calculator. When you put a number, called an input, into the input hopper of a function machine, the machine uses a mathematical rule to change that number into another one. The new number, called the output, comes out of the machine’s output spout. Here are two examples:

The first machine is called a “times 3” or a “× 3” machine; the second is called a “plus 2” or a “+ 2” machine.

Think about this:
What would come from the output spout of a × 3 machine if the input was 3? What if the input was 10? 1? 0? -5? 1.5? 1/3? Record the results in an input-output table like this:

<table>
<thead>
<tr>
<th>× 3 Machine</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>-5</td>
</tr>
<tr>
<td>1.5</td>
</tr>
<tr>
<td>1/3</td>
</tr>
</tbody>
</table>

Try some inputs for a + 2 machine as well, and record the outputs in a table. Invent several more simple function machines. Make a table to record some inputs and outputs for each one.
Function Machines—Combined Machines

It’s possible to make a combined function machine by connecting the output spout of one machine to the input spout of another. The output of the first machine becomes the input of the next machine. This is called a \([\times 3 + 2]\) machine.

Think about this:
What would happen if you put a 3 into the input hopper of this machine? Try 10, 1, 0, –5, 1.5, and 1/3 as inputs, and record the results in an input-output chart.

Now make a different combined machine by reversing the order of the component machines, creating a \([+ 2 \times 3]\) machine.

Think about this:
Would the results for a \([+ 2 \times 3]\) machine be the same as for your \([\times 3 + 2]\) machine? Try some inputs and see. What would happen if you put a 3 into the input hopper of this machine? Try 10, 1, 0, –5, 1.5, and 1/3 as inputs, and record the results in an input-output chart.

Invent some combined machines of your own. Make some two-machine and three-machine combos. Make an input-output chart for each machine.
Function Machines—Mystery Machines

Suppose you had a mystery machine—a function machine without a label. The only way you can figure out what the machine does is to test it with some inputs and see what the outputs are. Then, if you could build a machine that has exactly the same inputs and outputs as the mystery machine, you could give it a name.

Here is a set of inputs and outputs for a mystery machine:

<table>
<thead>
<tr>
<th>Mystery Machine</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
<td><strong>Output</strong></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>–5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>8.5</td>
<td></td>
</tr>
<tr>
<td>1/3</td>
<td>7 1/3</td>
<td></td>
</tr>
</tbody>
</table>

It looks like this could be a [+ 7] machine. But think again—you don’t know what’s inside. It might also be a [+ 3 + 4], a [– 3 + 10], or a [× 3 + 21 ÷ 3] machine. Since all these machines have the same results—the same input-output table—they are called equivalent machines.

Can you build a combined machine that has the same input-output table?

<table>
<thead>
<tr>
<th>Mystery Machine</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
<td><strong>Output</strong></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>–5</td>
<td>–6</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>1/3</td>
<td>4 2/3</td>
<td></td>
</tr>
</tbody>
</table>
## Number Trick Cards

### Group 1

<table>
<thead>
<tr>
<th>Card 1</th>
<th>Card 2</th>
<th>Card 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>—Think of a number.</td>
<td>—Think of a number bigger than 10.</td>
<td>—Think of a number.</td>
</tr>
<tr>
<td>‣ Add 3.</td>
<td>‣ Take half of it.</td>
<td>‣ Subtract 7.</td>
</tr>
<tr>
<td>‣ Double what you get.</td>
<td>‣ Subtract 5.</td>
<td>‣ Multiply by 3.</td>
</tr>
<tr>
<td>‣ Subtract 6.</td>
<td>‣ Multiply by four.</td>
<td>‣ Add 6.</td>
</tr>
<tr>
<td>‣ Divide the result by 2.</td>
<td>‣ Add 20.</td>
<td>‣ Take 1/3 of this number.</td>
</tr>
<tr>
<td>‣ What did you get?</td>
<td>‣ Take half of the result.</td>
<td>‣ Add 5.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>‣ What did you get?</td>
</tr>
</tbody>
</table>

1. **Other Representations:**
   - **Function Machine**
     
     1. \[+ 3 \times 2 - 6 \div 2\]
   - **Equation**
     \[y = \frac{2(x + 3) - 6}{2}\]
     \[= \frac{2x + 6 - 6}{2}\]
     \[= x\]

2. **Other Representations:**
   - **Function Machine**
     \[x \left(\frac{1}{2}\right) - 5 \times 4 + 20 \times \left(\frac{1}{2}\right)\]
   - **Equation**
     \[y = \left(\frac{1}{2}\right)[4(x + 2 - 5) + 20]\]
     \[= \left(\frac{1}{2}\right)[2x - 20 + 20]\]
     \[= x\]

3. **Other Representations:**
   - **Function Machine**
     \[- 7 \times 3 + 6 \times \left(\frac{1}{3}\right) + 5\]
   - **Equation**
     \[y = \left(\frac{1}{3}\right)[3(x - 7) + 6] + 5\]
     \[= \left(\frac{1}{3}\right)[3x - 21 + 6] + 5\]
     \[= x-5 + 5 = x\]

### Group 2

<table>
<thead>
<tr>
<th>Card 4</th>
<th>Card 5</th>
<th>Card 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>—Think of a number.</td>
<td>—Think of a number.</td>
<td>—Think of a Number that is a multiple of 3.</td>
</tr>
<tr>
<td>‣ Triple it.</td>
<td>‣ Double it.</td>
<td>‣ Divide it by 3.</td>
</tr>
<tr>
<td>‣ Take 1/3 of the result.</td>
<td>‣ Divide the result by 2.</td>
<td>‣ Triple your result.</td>
</tr>
<tr>
<td>‣ Subtract your original number.</td>
<td>‣ Add 4.</td>
<td>‣ Subtract your original number.</td>
</tr>
<tr>
<td>‣ Add 5.</td>
<td>‣ Subtract your original number.</td>
<td>‣ Add 7.</td>
</tr>
<tr>
<td>‣ What did you get?</td>
<td>‣ What did you get?</td>
<td>‣ What did you get?</td>
</tr>
</tbody>
</table>

4. **Equation**
   \[y = (3x + 6)(1/3) - x + 5\]
   \[= x + 2 - x + 5 = 7\]

5. **Equation**
   \[y = (2x - 8) \div 2 + 4 - x\]
   \[= x - 4 + 4 - x = 0\]

6. **Equation**
   \[y = 3(x - 3 - 2) - x + 7\]
   \[= x - 6 - x + 7 = 1\]
Realistic Situations

Part 1: Mowing Lawns

<table>
<thead>
<tr>
<th>Number of Lawns that Matt Mowed</th>
<th>Time that Matt Spent Mowing Lawns</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 lawn</td>
<td>30 minutes</td>
</tr>
<tr>
<td>5 lawns</td>
<td>150 minutes</td>
</tr>
<tr>
<td>6 lawns</td>
<td>180 minutes</td>
</tr>
<tr>
<td>8 lawns</td>
<td>4 hours</td>
</tr>
<tr>
<td>12 lawns</td>
<td></td>
</tr>
<tr>
<td>17 lawns</td>
<td></td>
</tr>
</tbody>
</table>

1. Write a mathematical rule to describe the relationship between time spent mowing and the number of lawns mowed.

2. Graph the relationship between time spent mowing and the number of lawns mowed.

Part 2: Candy Bars

# of Remaining Candy Bars

Days of Sale

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

500 450 400 350 300 250 200 150 100 50
1. Explain what the different parts of the graph mean.

2. Make an input-output chart for this linear relationship.

3. Write a mathematical rule for the relationship between the number of candy bars left to sell and the number of days that have gone by.

Part 3: Buying Bagels

You have gone to the store to buy bagels and cream cheese. You need one container of cream cheese for the bagels, which will cost $2.50, and you need 50 cents for each bagel you decide to buy. How would you figure out the total amount your order will cost?

1. Write a mathematical rule to represent the situation.

2. Make an input-output chart for this situation.

3. Draw a graph of the relationship between the number of bagels purchased and the total cost of the order.
Golden Apples

A prince picked some golden apples from an enchanted tree in an enchanted garden, and started to take them home. Before he could leave the garden he encountered a troll, who demanded that the prince give him half of all the apples he had, plus two more, as payment for being allowed to pass by. After traveling a few hundred yards farther he encountered a second troll, who demanded the same payment: half of all the apples he had with him, plus two more. Finally, he encountered a third troll, who also demanded half of all the apples he had with him, plus two more. After paying off the third troll, the prince made his way home without further difficulty—but he had only two golden apples left.

1. How many did he pick from the enchanted tree?

2. Suppose the prince came home with $N$ golden apples? How many did he pick?

3. Could the prince pick any number of golden apples and still come home with a whole number of apples? If not, what are the possible numbers of apples the prince could have picked?