Counting with tournaments
Problem

You know that athletic events can be a lot of fun. But they can also help solve math problems! Let's see how it works.

Some tennis tournaments are played using a "round robin" system: everybody plays everybody.

Winning more games than the others makes you the tournament winner.

1. How many games are played *by each person* if the number of participants is
   (a) 3?
   (b) 4?
   (c) 11?
   (d) n?

2. How many games are played *altogether* if the number of participants is
   (a) 3?
   (b) 4?
   (c) 11?
   (d) n?

3. The total number of games must be a whole number. Prove that the formula obtained in problem 2, part (d), always yields whole numbers.

4. Consider n points. Each point is connected by a segment to all others. How many segments are there?

5. How many diagonals does a regular n-gon have?

Use the result of problem 4.

Hints

There are no hints for this problem sequence.

Answers

1. (a) 2
(b) 3
(c) 10
(d) $n-1$

2. (a) 3
(b) 6
(c) 55
(d) $\frac{n(n-1)}{2}$

3. See solution.

4. $\frac{n(n-1)}{2}$

5. $\frac{n(n-3)}{2}$

Solutions

1. If there are 3 participants, say Abby, Ben, and Carlos, then each will play 2 games. For example, Abby plays Ben and Abby plays Carlos. Since each participant plays everyone else (except himself or herself), each must play $n-1$ games.

2. Here are two possible ways to approach this problem. One is to notice that each of the n players plays $(n-1)$ games, which gives us $n(n-1)$ games. But when Abby plays Ben, Ben plays Abby—and this is just one game, not two. Therefore, the answer is not $n(n-1)$, but $\frac{n(n-1)}{2}$.
Alternatively, in a two-player tournament, there is only 1 game. If one more player joins the tournament (for the total of 3 players), that player must play all others, adding 2 more games to the total. If one more player joins in, bringing the total number of players up to 4, it adds 3 more games, and so on. The nth player must play (and adds to the tournament) $(n-1)$ more games. Therefore, in a tournament with n players, there would be $1+2+\cdots+(n-1)=\frac{n(n-1)}{2}$ games.

3. Either n or $(n-1)$ must be an even number, so $n(n-1)$ is even; that is, divisible by 2 without a remainder—which means the answer *is* a whole number.

4. Imagine that the n participants in the round-robin tournament are standing on the points, one person for each point. The segments connecting two points could then represent a game played between the two people at those points. The total number of segments is equal to the total number of games played: $\frac{n(n-1)}{2}$.

5. The diagonals of an n-gon are the segments connecting each point to each *non-adjacent* point. A segment connecting two adjacent points is a side of the n-gon. Since there are $\frac{n(n-1)}{2}$ segments connecting each point to every other point, and n of those are not diagonals, there are $\frac{n(n-1)}{2}-n$ diagonals. This can be simplified to $\frac{n(n-3)}{2}$ diagonals.

