Strangely defined linear function
Problem

Often, a linear function $y = kx + l$ is given by its slope k and y-intercept l. For any pair of values of k and l, there is exactly one corresponding function $y(x)$.

You may be familiar with function notation $f(x) = mx + b$.

Of course, that way of defining a linear function is not the only one possible. Here is another.

Imagine, instead, that the two values you know are the x-intercept (let us call it x_0) and the distance (call it d) between the x- and the y-intercept.

For these problems, let both x_0 and d be *non-zero* numbers. Are the answers different if one or both of these are zero?

1. How many different linear functions are defined by a given pair x_0 and d?
Look at these three cases:
- $d < |x_0|$
- $d = |x_0|$
- $d > |x_0|$

2. If $d = 10$ and $x_0 = 6$, find the slope and the y-intercept of both functions defined by d and x_0 and write these functions in the $(y = kx + l)$ form.

3. Now consider a more general case. Two functions are defined by a pair of numbers d and x_0 $(d > x_0 > 0)$. Find the slope and the y-intercept of each function in terms of d and x_0. Write each function as $y = kx + l$.

No numerical values are given this time.

4. Let us revisit Problem 3. If $d = 2x_0$, write the expressions for both functions found there.

These expressions will not be quite as long...

Hints

Hint to problem 1. On a graph, consider a triangle with the sides $d, |x_0|$ and $|y_0|$.

Hint to problem 2. Use the Pythagorean theorem to find $|y_0|$.

Answers

1.
- For $d < |x_0|$: None.
- For $d = |x_0|$: One; $y = 0$.
- For $d > |x_0|$: Two.

2. The y-intercept is either -8 or 8; the corresponding slopes are $\frac{4}{3}$ and $-\frac{4}{3}$. The functions are:
$y = \frac{4}{3}x - 8$ and $y = -\frac{4}{3}x + 8$.

3. The y-intercept is either $\sqrt{d^2 - (x_0)^2}$ or $-\sqrt{d^2 - (x_0)^2}$; the slopes are $-\frac{\sqrt{d^2-(x_0)^2}}{x_0}$ and $\frac{\sqrt{d^2-(x_0)^2}}{x_0}$, respectively.
The two functions are:

$y = -\frac{\sqrt{d^2-(x_0)^2}}{x_0}x + \sqrt{d^2 - (x_0)^2}$
and
$y = \frac{\sqrt{d^2-(x_0)^2}}{x_0}x - \sqrt{d^2 - (x_0)^2}$.

4. $y = -\sqrt{3}x + x_0\sqrt{3}$ and $y = \sqrt{3}x - x_0\sqrt{3}$

Solutions

1.
- $d < |x_0|$: None; d is the hypotenuse in the right triangle with the sides $|x_0|, d$, and $|y_0|$. Therefore, it can not be smaller than $|x_0|$.
- $d = |x_0|$: One; since the y-intercept has to be zero (why?), this function is $y = 0$
- $d > |x_0|$: Two; the y-intercept has two possible locations: one positive, and one negative.

2. Let's use the Pythagorean theorem to determine $|y_0|$. $|y_0| = \sqrt{d^2 - (x_0)^2} = \sqrt{(10)^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8$. 8 is the distance from the origin to y_0 along the y-axis, therefore the y-intercept is either -8 or 8. The slope is determined as $-\frac{y_0}{x_0}$, so the corresponding slopes are $\frac{4}{3}$ and $-\frac{4}{3}$. The functions are:
$y = \frac{4}{3}x - 8$ and $y = -\frac{4}{3}x + 8$.

3. Again, use the Pythagorean theorem to find $|y_0|$. The y-intercept is either $\sqrt{d^2 - (x_0)^2}$ or $-\sqrt{d^2 - (x_0)^2}$; the slopes are $-\frac{\sqrt{d^2-(x_0)^2}}{x_0}$ and $\frac{\sqrt{d^2-(x_0)^2}}{x_0}$, respectively.
The two functions are:

$y = -\frac{\sqrt{d^2-(x_0)^2}}{x_0}x + \sqrt{d^2 - (x_0)^2}$
and
$y = \frac{\sqrt{d^2-(x_0)^2}}{x_0}x - \sqrt{d^2 - (x_0)^2}$.

4. In the answers for the previous problem, substitute d for $2x_0$.
$y = -\frac{\sqrt{d^2-(x_0)^2}}{x_0}x + \sqrt{d^2 - (x_0)^2} =$
$y = -\frac{\sqrt{(2x_0)^2-(x_0)^2}}{x_0}x + \sqrt{(2x_0)^2 - (x_0)^2} =$
$y = -\frac{\sqrt{3x_0^2}}{x_0}x + \sqrt{3x_0^2}$
$y = -\sqrt{3}x + x_0\sqrt{3}$
and $y = \frac{\sqrt{d^2-(x_0)^2}}{x_0}x - \sqrt{d^2 - (x_0)^2} =$
$y = \frac{\sqrt{(2x_0)^2-(x_0)^2}}{x_0}x - \sqrt{(2x_0)^2 - (x_0)^2} =$
$y = \frac{\sqrt{3x_0^2}}{x_0}x - \sqrt{3x_0^2}$
$y = \sqrt{3}x - x_0\sqrt{3}$

