

Commutativity and associativity

Inspired by the Nuffield Mathematics Project (John Wiley and Sons)

Addition, subtraction, multiplication, and division are all mathematical *operations*. With each, you take two numbers and get a single number from them. The kind of number you get depends on the symbol between them: $4 + 2$ is a different number from $4 \div 2$.

- Here is a different kind of operation, with a few examples:

$$\begin{array}{lll} 5 \diamond 4 = 5 & 5 \diamond 8 = 8 & 3 \diamond 6 = 6 \\ 9 \diamond 9 = 9 & 23 \diamond 12 = 23 & 0 \diamond 19 = 19 \end{array}$$

- What do you think is the value of $19 \diamond 3$? Describe the rule you used to get your answer.
- Does it matter which number comes before the symbol \diamond ? If a and b represent any two numbers, is it true that $a \diamond b = b \diamond a$?
- Pick any three numbers. Call your numbers a , b , and c . Is the following true?

$$(a \diamond b) \diamond c = a \diamond (b \diamond c)$$

- Is $(a \diamond b) \diamond c = a \diamond (b \diamond c)$ for any three numbers you choose?

When the order doesn't matter, the operation is **commutative**. *Commute* means travel. You can make the numbers on each side of the symbol travel across the symbol without changing the result. When the grouping doesn't matter, the operation is **associative**. *Association* means group. You can group the numbers in different ways without changing the result.

Find a rule for each operation, and decide if the operation is commutative or associative.

- $$\begin{array}{lll} 5 \spadesuit 4 = 54 & 5 \spadesuit 8 = 58 & 3 \spadesuit 6 = 36 \\ 0 \spadesuit 9 = 9 & 2 \spadesuit 12 = 32 & 19 \spadesuit 73 = 263 \end{array}$$

- $$\begin{array}{lll} 5 \heartsuit 4 = 1 & 5 \heartsuit 8 = 3 & 3 \heartsuit 6 = 3 \\ 9 \heartsuit 9 = 0 & 23 \heartsuit 12 = 11 & 0 \heartsuit 19 = 19 \end{array}$$

- $$\begin{array}{lll} 5 \clubsuit 4 = 9 & 5 \clubsuit 8 = 13 & 3 \clubsuit 6 = 9 \\ 9 \clubsuit 9 = 18 & 23 \clubsuit 12 = 35 & 0 \clubsuit 19 = 19 \end{array}$$

- Which of the standard operations (addition, subtraction, multiplication, and division) are commutative? Which are associative?

Answers

1. (a) $19 \diamond 3 = 19$. The result is whichever number is greater. If the numbers are equal, the result is the same as the other numbers.
- (b) It doesn't matter which number is first and which is second.
- (c) For any choice of a , b , and c ,

$$(a \diamond b) \diamond c = a \diamond (b \diamond c).$$

- (d) Yes.
2. Multiply the first number by 10 and add the second number ($a \spadesuit b = 10a + b$). This is neither commutative nor associative.
3. Subtract the larger from the smaller. This is commutative but not associative.
4. Add the two numbers together. This is both commutative and associative.
5. Addition and multiplication are both commutative and associative, but subtraction and division are neither.