Domain Event
Problem

1. The following functions use real numbers for both the inputs and the outputs. Try to evaluate them for inputs -3, 0, and 3.

(a) $f(a) = 3a^2 + 1$
(b) $g(d) = \frac{12}{d}$
(c) $y = \frac{8x^2}{x}$
(d) $q(r) = \sqrt{2 - r}$

2. Think about the functions in problem 1 which couldn't be evaluated at one or more inputs. For each, are there any other inputs that can't be used? Explain.

3. The **domain** of a function is the set of numbers that can be used as inputs. Although problem 1 stated that the inputs were real numbers, the form of a function can limit its domain to a subset of the stated set. Describe the domains for all four functions in problem 1.

4. For each function below, describe its domain as a subset of the real numbers.

(a) $f(x) = (x - 2)(x + 1)$
(b) $b(t) = \frac{3t}{t}$
(c) $c(t) = \frac{3t}{t-2}$
(d) $d(t) = \frac{3t}{3(t-2)(t+1)}$
(e) $y = \sqrt{x}$
(f) $s = \sqrt{r + 18}$
(g) $g(h) = \sqrt{h^2}$
(h) $i(d) = \sqrt{h^2 - 5}$
(i) $s(n) = \frac{\sqrt{(n+1)(n-3)}}{n+9}$

5. Another way to restrict a function is to state the domain: Let f be a function such that $f(x) = \frac{3}{x} - 3$, where x is a real number and $x < 6$.

(a) Is 10 in the domain of f?
(b) Is 3 in the domain of f?
(c) Is 0 in the domain of f?
(d) Is -3 in the domain of f?

6. Serena has put a can outside to catch rain. It is 8 inches tall, with straight sides. She uses it to measure how much rain she gets. At the beginning of one moderate rainstorm, the can held an inch of water. The depth d of the rain in the can was related to the time t since the storm began by the function $d(t) = 0.2t + 1$, where t is measured in hours and d is measured in inches.

(a) First, consider only the function $d(t) = 0.2t + 1$, and not what the variables mean. What would you say is the domain of this function?

(b) Now think about what lengths of time can be considered for the situation. What limitations does this impose on the domain of the function?

(c) What depths of rain are possible for the can, in this storm? Are there times for which the function gives incorrect values?

(d) In the context of the problem situation, what would you say is the domain of the function $d(t) = 0.2t + 1$? That is, for what values of t do both t and the corresponding d make sense?

(e) A good practice when writing a function to describe a situation is to state the domain when the function is stated. (This helps people realize what input values give function values that will be accurate in the situation.) Find the sentence where the function was stated, and rewrite it to include a statement of the domain.

7. Describe the three ways to restrict domains that were presented in these problems. Use the problems as examples in your descriptions.

Hints

1. For part (c), do not simplify the definition expression before you try to evaluate. The function was defined that way (instead of in the simplified way) for a reason.

2. Think about why a certain input caused the evaluation to go wrong. Are there other inputs that could cause the same problem?

3. You might say something like, "x is a real number, except x cannot. . . ," or "x is a real number and x must be. . . ."

4. Think about the kinds of things that went wrong in problem 1. For these functions, what inputs (if any) would make the same kind of things go wrong?
In part (d), there are two real numbers that must be excluded from the domain.
In parts (g) and (h), you might want to check your answers by trying to evaluate the function using any numbers you would exclude.
Be careful with part (i). You might try several example inputs, or graph at least the numerator of the radicand.

The **radicand** is the number or expression under the radical, in this case the square root sign.

5. The domain statement can tell you immediately that a particular number is not in the domain, but you also need to check the form of the function to be sure there aren't other restrictions.

6. (a) Imagine that the function was given in problem 1. What would you have answered then?
(b) What is the least possible value for t? Is there a greatest possible value?
(c) What is the least possible value for the depth? Is there a greatest value? What are the corresponding values for t?
(d) Pull all your observations into one statement about what numbers are allowable.
(e) Look at how problem 5 was worded and do something similar. When the variables' meaning is explained might be a good place to include the domain statement.

If you want, you might try writing two sentences instead of just one. The second sentence might start, "For this function. . . ."

7. Briefly said, the three ways can be described as *practical, implicit,* and *explicit.* Identify which problems show which type, and explain what each term means in relation to the restrictions for these problems.

Answers

1. (a) $f(-3) = 28$; $f(0) = 1$; $f(3) = 28$
(b) $g(-3) = -4$; $g(0)$ is undefined; $g(3) = 4$
(c) If $x = -3$, $y = -24$; if $x = 0$, y is undefined; if $x = 3$, $y = 24$.
(d) $q(-3) = \sqrt{5}$; $q(0) = \sqrt{2}$; $q(3)$ is undefined.

2. For parts (b) and (c), no; for part (d), yes. See the solutions for an explanation.

3. (a) all real numbers
(b) all real numbers except 0
(c) all real numbers except 0
(d) all real numbers less than or equal to 2

4. (a) all real numbers
(b) all real numbers except 0
(c) all real numbers except 2
(d) all real numbers except 2 and -1
(e) all nonnegative real numbers (that is, all real numbers greater than or equal to 0)
(f) all real numbers greater than or equal to -18
(g) all real numbers
(h) all real numbers greater than or equal to 5
(i) all real numbers greater than or equal to 3, or less than or equal to -1, except -9

A shorter way to say this is "all real numbers n such that $n \geq 3$ or $n \leq -1$, and $n \neq -9$."

5. (a) no
(b) yes
(c) no
(d) yes

6. (a) all real numbers
(b) $t \geq 0$; upper limits for t may vary.
(c) 1 inch to 8 inches; yes
(d) $0 \leq t \leq 35$
(e) See solutions.

7. See solutions.

Solutions

1. (a) $f(-3) = 3(-3)^2 + 1 = 3(9) + 1 = 28$;
$f(0) = 3(0)^2 + 1 = 1$;
$f(3) = 3(3)^2 + 1 = 28$

(b) $g(-3) = \frac{12}{-3} = -4$;
$g(0) = \frac{12}{0}$, which is undefined; $g(3) = \frac{12}{3} = 4$

(c) When $x = -3$, $y = \frac{8(-3)^2}{-3} = 8(-3) = -24$;
when $x = 0$, $y = \frac{8(0)^2}{0}$, which is undefined;
when $x = 3$, $y = \frac{8(3)^2}{3} = 24$.

(d) $q(-3) = \sqrt{2-(-3)} = \sqrt{2+3} = \sqrt{5}$;
$q(0) = \sqrt{2-0} = \sqrt{2}$; $q(3) = \sqrt{2-3} = \sqrt{-2}$, which is undefined.

2. The functions in parts (b) and (c) are undefined only when the denominator is 0, so there's only one input (0 for each) that can't be used. The function in part (d) is undefined any time the radicand is negative, so there are several inputs (for example, 5, 9.3, and 1,000) that can't be used.

The **radicand** is the number or expression under the radical, in this case the square root sign.

3. (a) Any real number will work, so the domain is all real numbers.

(b) Only $d = 0$ will make the denominator 0, so the domain is all real numbers except 0.

(c) Again, only $x = 0$ will make the denominator 0. The domain is all real numbers except 0.

(d) The radicand is $2 - r$. This is 0 when $r = 2$, and negative only when r is greater than 2. So the domain is all real numbers greater than 2.

4. (a) Adding or subtracting two real numbers always gives another real number. Similarly, multiplying two real numbers always gives another real number. So there are no restrictions to the domain for this function.

(b) The denominator is t; since denominators can't be 0, the domain is all real numbers except 0.

(c) The denominator is $t-2$. This is 0 when $t = 2$, so the domain is all real numbers except 2.

(d) The denominator for this function is $3(t-2)(t+1)$, a product of three numbers. Thie product can be 0 whenever one of the factors is 0. Since 3 can never be 0, find what values of t make $t-2$ and $t+1$ equal to 0. The domain is all real numbers except 2 and -1.

(e) The radicand, x, cannot be negative, so the domain is all real numbers greater than or equal to 0.

(f) For $r + 18$ to be greater than or equal to 0, r must be greater than or equal to -18.

(g) Again, the radicand has to be nonnegative. Since a number multiplied by itself is always positive or 0, the expression under the radical is never negative. All real numbers are in the domain of this function.

(h) For this function, however, the radicand can be negative. The domain is the solution to the inequality $h^2 - 5 \geq 0$, which is $h \geq \sqrt{5}$. Another way to think of this is to recognize that h^2 must be greater than 5 for $h^2 - 5$ to be positive, so h must be greater than or equal to $\sqrt{5}$ for the expression to be nonnegative.

(i) There are two things to consider here, the square root in the numerator and the expression in the denominator. Starting with the numerator: the expression $(n + 1)(n - 3)$ is negative when one factor is negative and the other is positive. There are several ways to solve this. For example, you might find when the expression is 0 (when $n = -1$ or $n = 3$), graph those on a number line, and test values in the intervals between and on either side of the values. The expression is negative between the two values, so from the numerator we can restrict the domain to real numbers greater than or equal to 3 and less than or equal to -1. However, the denominator can't be 0, so -9 must be excluded from that domain.

The double threat!

5. (a) The stated restriction is that $x < 6$. Since 10 is not less than 6, 10 is not in the domain.

(b) The stated restriction doesn't exclude 3 (because $3 < 6$). The denominator of the fraction is not 0 when $x = 3$, and there are no other reasons to restrict the domain, so 3 is in the domain.

(c) Although $0 < 6$, 0 makes the denominator equal to 0, so 0 is not in the domain.

(d) Since $-3 < 0$ and -3 does not make the denominator 0, -3 is in the domain.

6. (a) No real numbers would cause trouble evaluating this function. Removed from context, then, the domain is all real numbers.

(b) Negative values for time don't really make sense. One interpretation for negative values of time is time before the moment chosen as $t = 0$, in this case before the storm started. However, because the function decribes the depth of the rainwater as the can fills, time before the storm started doesn't matter—the rain wasn't filling the can then.

In terms of how long the storm might last, the greatest value that t might take can't really be determined. Some storms can last for a day or more.

(c) At the beginning of the storm, the can has 1 inch of rainwater; when it's completely full, it will have 8 inches of rain in it. Once the can is full, it overflows—it can't hold 9 inches of water, for example. Any time values that make the function value less than 1 or more than 8 will give incorrect information.

(d) The can has 1 inch of water immediately (and for some time) before the storm started ($t = 1$). Evaluating the function with a negative value (meaning time in hours before the storm started) will give a value less than 1, so these values will not be correct. So $t \geq 0$.

The can will be full when it has 8 inches of water in it, that is, at the time t when $d(t) = 8$. Solving $0.2t + 1 = 8$ gives $t = 35$, so the can will be full after 35 hours of rain. So the domain is $0 \leq t \leq 35$, assuming that the storm lasts that long. (Otherwise the upper time limit is when the storm ends.)

This function assumes that the rate at which the rain falls is constant.

(e) The sentence defining the function is "The depth d of the rain in the can was related to the time t since the storm began by the function $d(t) = 0.2t + 1$, where t is measured in hours and d is measured in inches."

One way to rewrite this is, "The depth d of the rain in the can was related to the time t since the storm began by the function $d(t) = 0.2t + 1$, where t is measured in hours, $0 \leq t \leq 35$, and d is measured in inches."

7. Problems 1 and 4 show *implicit* ways a domain can be restricted. The form of the function excludes certain values as inputs. Problem 5 shows an *explicit* way to restrict the domain: you tell what values are allowed. (Implicit restrictions can still apply.) Problem 6 shows how a function might have *practical* restrictions: within the context of the situation, some values of the input, or the corresponding values of the output, might not make sense.

