And multiples all in a row

by Gil French

1. Consider any set of 7 consecutive numbers.

(a) Does the set contain a multiple of 7? Do you think any set of 7 consecutive numbers will contain a multiple of 7?

(b) Does the set contain a second multiple of 7? Do you think any set of 7 consecutive numbers will contain two multiples of 7?

2. Consider any set of 9 consecutive numbers.

(a) Does the set contain a multiple of 9? Do you think any set of 9 consecutive numbers will contain a multiple of 9?

(b) Does the set contain a second multiple of 9? Do you think any set of 9 consecutive numbers will contain two multiples of 9?

3. Consider any set of 5 consecutive numbers.

(a) Does the set contain a multiple of 5? Do you think any set of 5 consecutive numbers will contain a multiple of 5?

(b) Does the set contain a second multiple of 5? Do you think any set of 5 consecutive numbers will contain two multiples of 5?

4. Make a general statement about finding a multiple of a given number (call it n) in that many consecutive numbers. (That is, how many multiples of n are there in n consecutive numbers?) Explain how you know your statement is true.

Just saying that it worked for a few cases is not enough! You need to say why you know it will be true for every case.

Here are the numbers from 1 to 100, arranged in ten columns:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

5. Consider *vertical* sets of numbers, such as $\{23, 33, 43, 53\}$.

(a) For a set of 7 consecutive numbers in a column, will you always find a multiple of 7? Will you ever find more than one multiple of 7?

(b) Look for multiples of 9 in sets of 9, and for multiples of 5 in sets of 5.

(c) Investigate other set sizes, such as 2, 3, 4, and 10.

(d) Make a conjecture about finding multiples of n in a set of n consecutive numbers in a column. Prove that your conjecture is true.

6. Now consider *diagonal* sets of numbers, such as the set $\{32, 43, 54, 65, 76\}$. Make a conjecture about finding multiples of n in a diagonal set of n numbers.

7. Could a different arrangement of numbers give more multiples of 7 in a vertical or diagonal set? Explain.

Hints

Hint for problem 4. What are the remainders of 7 consecutive numbers when you divide by 7? What are the remainders of n consecutive numbers when you divide by n? What is the remainder if the number is a multiple of n?

Hint for problem 5d. Look at the greatest common divisor of n and 10 (the difference between two consecutive vertical numbers). For a set of n consecutive vertical numbers in which there isn't exactly one multiple of n, is there a relationship between the GCD and the largest number of multiples you can find in the set?

Hint for problem 7. Try different arrangements and see for yourself. When deciding what arrangement to try, look at the given one (rows of 10). How does how many you put in a row change what number goes below another?

Answers

1. (a) Any set of 7 consecutive numbers will contain a multiple of 7.

(b) No set of 7 consecutive numbers will contain two multiples of 7.

2. (a) A set of 9 consecutive numbers will contain a multiple of 9.

(b) No set of 9 consecutive numbers will contain two multiples of 9.

3. (a) A set of 5 consecutive numbers will contain a multiple of 5.

(b) No set of 5 consecutive numbers will contain two multiples of 5.

4. Any set of n consecutive numbers will have exactly one multiple of n. See the solutions for a full explanation.

5. (a) There will always be exactly one multiple of 7.

(b) There will always be exactly one multiple of 9. However, for multiples of 5, either there is no multiple or they are all multiples.

(c) Sets of 2 work like sets of 5; either there is no multiple or they are all multiples. Sets of 3 have exactly one multiple of 3. Sets of 4 have either no multiples of 4 or *2* multiples of 4.

(d) If the greatest common divisor (GCD) of n and 10 is 1 (that is, if n doesn't have a factor of 2 or 5), then there is exactly one multiple of n in the set of n consecutive vertical numbers. If the GCD is not 1, then there will be either no multiples of n or a number of multiples equal to the GCD.

For example, the GCD of 8 and 10 is 2, so there will be either no multiples of 8 or 2 multiples of 8 in any set of 8 consecutive vertical numbers.

6. The difference is this case is 11, which is prime, so if n isn't a multiple of 11, the set will have exactly one multiple of 11. If n is a multiple of 11, the set will have either no multiples of 11 or 11 multiples of 11.

7. Yes, a different arrangement would matter. Arranging by 7s would allow either no multiples of 7 or 7 multiples of 7, for any set whose size is itself a multiple of 7. Arranging by one less than a multiple of 7 (for example 6, 13, or 20) would give diagonal numbers that differ by 7, so a set of diagonal numbers could have no multiples or 7 multiples.

For example, arranging by 7s and then choosing a set of 7 consecutive vertical numbers.

Solutions

Solution for problem 4. Think about dividing the first number in a set of 7 numbers by 7. If it's not a multiple of 7, there will be a remainder. The next consecutive number will have a remainder of one more, unless the first number's remainder was 6. In that case, the remainder would have been 7, but the remainder has to be less than the divisor, so it's actually 7 less, or 0. So the second number would be a multiple of 7. (For example, $13 \div 7$ is 1 remainder 6, so 14 is a multiple of 7.) There are only 7 possible remainders (including 0 or no remainder), and the 7 numbers will each have a different remainder. So one of them will have to have no remainder (remainder 0), which means that number is a multiple of 7.

More generally, if you divide by n, you must have a remainder from 0 to n (0 when the number is a multiple of n). Each consecutive number will have a remainder one greater, until the remainder is $n-1$. Then the next number will be a multiple of n. There can be only one multiple, because in n consecutive numbers, each of the n remainders from 0 to $n-1$ has to appear once.

