When perimeter equals area (right triangle)

There are only two triangles with whole number leg lengths for which the area and the perimeter are equal.

Don't count right triangles whose base and height are switched as different. For example, a right triangle whose base is 3 and height is 9 is the same rectangle as one whose base is 9 and height is 3.

1. Find the two triangles.
2. Prove that they are the only triangles for which this is possible.

Hints

Hint for problem 2. Set up an equation for a right triangle with legs b and h, and then solve for either variable. The result is a fraction; rewrite to get a whole number plus a fraction. (Find a value for k so that $\frac{ax+b}{x+c} = a + \frac{k}{x+c}$.) What kind of number must the fraction part ($\frac{k}{x+c}$) be? What does that tell you about the denominator?

Solutions

1. The sides of any such triangle must be Pythagorean triples. The area is the product of whole number leg lengths (divided by 2). The perimeter then must be a whole number (or half of a whole number), because it is equal to the area. For the perimeter to be half of a whole number, the hypotenuse must be half of a whole number. Of course, if a and b are whole numbers, $\sqrt{a^2+b^2}$ must either be a whole number or irrational, so for these triangles, all three side lengths must be whole numbers.

The most common examples for Pythagorean triples are $\{3, 4, 5\}$ and $\{5, 12, 13\}$; in fact, $\{6, 8, 10\}$ and $\{5, 12, 13\}$ give the correct triangles. The areas (and perimeters) are 24 and 30.

2. Let b and h be the legs of a right triangle. If the area is the same as the perimeter,

$$\begin{aligned}
\frac{1}{2}bh &= b+h+\sqrt{b^2+h^2} \\
\frac{1}{2}bh-b-h &= \sqrt{b^2+h^2} \\
\frac{1}{4}b^2h^2-b^2h-bh^2+2bh+b^2+h^2 &= b^2+h^2 \\
\frac{1}{4}b^2h^2-b^2h-bh^2+2bh &= 0 \\
bh-4b-4h+8 &= 0 \\
b(h-4) &= 4h-8 \\
b &= \frac{4h-8}{h-4} \\
b &= 4+\frac{8}{h-4}
\end{aligned}$$

Since b is a whole number, 8 must be divisible by $h-4$, which means $h-4=1$, $h-4=2$, $h-4=4$, or $h-4=8$. That means h must be 5, 6, 8, or 12, and b is respectively 12, 8, 6, or 5. The legs are 12 and 5 or 6 and 8, giving areas (and perimeters) of 30 and 24.

