i see
Problem

Adapted from *Mathematical Methods in High School,* EDC

1 Extending the Number System

1. Find two numbers whose sum is 10 and product is 24.

2. Now suppose you want two numbers whose sum is 10 and product is 20. The numbers both have a square root involved—no rational numbers will fit this description.

(a) Write an equation to help you find these numbers.

(b) Solve your equation and find the two numbers.

(c) Check the result: Do your numbers have a sum of 10? Do they have a product of 20?

Rational numbers can be written as $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

3. Even using square roots is sometimes not enough to solve problems of this type.

(a) Try to find two numbers whose sum is 10 and product is 30, using the method you used in problem 2.

(b) Why would someone say the problem in part (a) has no solution?

In order to solve problems like problem 2, the number system of rational numbers had to be expanded to include square roots of all positive rational numbers. Suppose you could invent a new number that would allow you to find two numbers whose sum is 10 and whose product is 30.

(c) What number would you invent?

(d) Find the two numbers.

(e) Can you show that your two numbers have a sum of 10 and a product of 30?

2 Getting precise

You may recall that if a and b are non-negative real numbers,

$$\sqrt{a}\sqrt{b} = \sqrt{ab}.$$

Assume that square roots of negative numbers exist. If the property above holds for negative numbers too, then

$$\sqrt{-9}\sqrt{-4} = \sqrt{(-9)(-4)} = \sqrt{36} = 6.$$

However, there's a problem. You can also do this:

$$\begin{aligned}
\sqrt{-9}\sqrt{-4} &= \sqrt{(9)(-1)}\sqrt{(4)(-1)} \\
&= (\sqrt{9}\sqrt{-1})(\sqrt{4}\sqrt{-1}) \\
&= 3(\sqrt{-1})(2)(\sqrt{-1}) \\
&= 6(\sqrt{-1})^2 \\
&= 6(-1) = -6
\end{aligned}$$

This makes it seem like $6 = -6$! Both numbers have 36 as their square, but people have agreed that $\sqrt{a}$ refers only to the non-negative square root of a. That means $\sqrt{36} = 6$, not -6. When you extend the real numbers to include numbers like $\sqrt{-9}$, you want the new numbers to work in the same way as the real numbers. To do that, though, you need to be sure that a calculation like $\sqrt{-9}\sqrt{-4}$ has only one interpretation.

The property that $\sqrt{a}\sqrt{b} = \sqrt{ab}$ when a and b are positive is helpful when working with square roots. For example, $\sqrt{54} = \sqrt{(9)(6)} = \sqrt{9}\sqrt{6} = 3\sqrt{6}$. The first step when working with a number like $\sqrt{-9}$ is to separate the 9 from -1: $\sqrt{-9} = \sqrt{(9)(-1)} = \sqrt{9}\sqrt{-1} = 3\sqrt{-1}$. In fact, all square roots of negative numbers can be separated into a square root of a positive number and $\sqrt{-1}$.

That means introducing only one new number, $\sqrt{-1}$, allows you to find square roots of all negative numbers. People who work with these numbers use the letter i to represent $\sqrt{-1}$.

1. Find the following, using i for $\sqrt{-1}$.
(a) $\sqrt{-9}$
(b) i^2
(c) i^3
(d) i^4

You now have two types of numbers: real numbers and numbers of the form bi, where b is a real number. For these numbers to form a number system, you have to be able to operate on them: add, subtract, multiply, and divide. Of course, you know how to operate with two real numbers.

2. For one of the following cases, you won't be able to apply the addition and multiplication rules you already know. Find the result for the other three cases.
(a) $3i + 2i$
(b) $3i \times 2i$
(c) $3 + 2i$
(d) $3 \times 2i$

3. You now have numbers of three types:
- a, where a is a real number
- bi, where b is a real number and i is $\sqrt{-1}$
- The combination of a and bi from problem 2 that couldn't be simplified $(a + bi)$

The last type actually includes the first two. How can you write -3 and $4i$ in that form?

$$-3 = \_\_ + \_\_i \qquad\qquad 4i = \_\_ + \_\_i$$

> A **complex number** is an expression of the form $a + bi$, where a and b are real numbers and $i^2 = -1$.

Once numbers are written in this form, you can add and multiply in the same way you would with real numbers such as $3+\sqrt{5}$.

4. Write each complex number in the form $a+bi$ where a and b are real numbers.
(a) $5 + 7\sqrt{-16}$
(b) $\sqrt{-16}$
(c) 67
(d) $\sqrt{16}$

5. Expand the following.

(a) $(x + 1)(x - 6)$
(b) $(3 + \sqrt{5})(2 - \sqrt{5})$
(c) $(1 + 7i)(2 - 2i)$

6. Calculate the following.

(a) $\sqrt{-9}\sqrt{-4}$
(b) $(3 + 2i) + (9 - i)$
(c) $(3 + 2i)(9 - i)$
(d) $(7 + 2i)(3 - 4i)$
(e) $(5 + 2i) + (5 - 2i)$
(f) $(5 + 2i)(5 - 2i)$
(g) $(a + bi) + (a - bi)$
(h) $(a + bi)(a - bi)$
(i) $(a - bi) + (c - di)$
(j) $(a - bi)(c - di)$

7. Using complex numbers, you now can find numbers with any sum and any product.

(a) Find two complex numbers whose sum is 10 and product is 30.
(b) Find two numbers whose sum is s and product is p, where s and p are real numbers.
(c) For what values of s and p are the numbers real?
(d) For what values of s and p are the numbers complex?

Hints

1 Extending the Number System

1. You should be able to find these numbers just by thinking about factors of 24.

2. (a) To write a single equation, you might start by writing two equations with two variables, x and y. Solve one equation for y and substitute the resulting expression in the other equation.

(b) Use the quadratic formula: For a quadratic equation in the form $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(c) Use the distributive property to find the product. For example, how would you expand $(1 + x)(2 - 3x)$?

3. (a) Write an equation, then use the quadratic formula.

(b) What causes a problem when you use the quadratic formula?

(c) What number, if it existed, would allow the result in part (b) to be acceptable?

(d) In part (c), you invented a number that made the result in part (b) acceptable.

(e) Assume that addition and multiplication work the same for your invented number as they do for a variable, say x. (If necessary, actually substitute x for your invented number, and then replace x with your number after you've completed your calculations.) You may need to think about what the square of your invented number would be.

2 Getting precise

1. (a) Begin by writing -9 as a product of -1 and a positive number.

(b) What does it mean for a number to be a square root of a number? For example, if $x = \sqrt{5}$, what is x^2?

(c) $i^3 = i(i^2)$
(d) Use one of your answers to parts (b) and (c).

2. Pretend i is a variable like x and try to simplify each expression. For example, how would you simplify $3x + 2x$?

3. What number you can multiply by i that will make that part of the term go away? (What do you think "go away" means?) If $bi = a + bi$, what value must a have?

4. Use the hints for problem 1, part (a) and problem 3.

5. Use the distributive property for each problem.

6. For part (a), write the numbers using i first. Once all numbers are written using i, you can multiply and add as if i were a variable. However, you know what real number i^2 is equal to, so your final expression should not have an i^2 term.

7. (a) Write an equation to solve this, as you did in section 1. Use the quadratic formula to solve the equation.
(b) Do the same as you did in part (a), using s for 10 and p for 30.
(c) Before you extended the real numbers, how did you decide that an equation had no solution?
(d) Remember, a complex number is any number that can be written in the form $a+bi$. Look back over problem 4.

Answers

1 Extending the Number System

1. 4 and 6

2. (a) Several answers are acceptable, but all should be equivalent to $x^2 - 10x + 20 = 0$.
(b) $5 + \sqrt{5}$ and $5 - \sqrt{5}$
(c) See solutions.

3. See solutions.

2 Getting precise

1. (a) $3i$
(b) -1
(c) $-i$
(d) 1

2. (a) $5i$
(b) -6
(c) $3 + 2i$ can't be combined.
(d) $6i$

3. $-3 = -3 + 0i$ and $4i = 0 + 4i$

4. (a) $5 + 28i$
(b) $0 + 4i$
(c) $67 + 0i$
(d) $4 + 0i$

5. (a) $x^2 - 5x - 6$
(b) $1 - \sqrt{5}$
(c) $16 - 12i$

6. (a) -6
(b) $12 + i$
(c) $29 + 15i$
(d) $29 - 22i$
(e) 10 (or equivalently, $10 + 0i$)
(f) 29 (or equivalently, $29 + 0i$)
(g) $2a$ (or equivalently, $2a + 0i$)

(h) $a^2 + b^2$ (or $a^2 + b^2 + 0i$)
(i) $a + c - (b + d)i$
(j) $ac - bd - (ad + bc)i$

7. (a) $5 + i\sqrt{5}$ and $5 - i\sqrt{5}$
(b) $\frac{s+\sqrt{s^2-4p}}{2}$ and $\frac{s-\sqrt{s^2-4p}}{2}$
(c) $s^2 - 4p \geq 0$, or equivalently, $s^2 \geq 4p$
(d) All values of s and p

Solutions

1 Extending the Number System

1. The factor pairs of 24 are 1 and 24, 2 and 12, 3 and 8, and 4 and 6. The sums of these pairs are (respectively) 25, 14, 11, and 10. So the numbers 4 and 6 fit the description.

2. (a) If the numbers are x and y, $x + y = 10$ and $xy = 20$. Solving the first for y gives $y = 10-x$. Substituting for y in the second then gives the following, all of which are acceptable answers:

Of course, other answers could be acceptable, too.

$$\begin{aligned} x(10-x) &= 20 \\ 10x - x^2 &= 20 \\ -x^2 + 10x - 20 &= 0 \\ x^2 - 10x + 20 &= 0 \end{aligned}$$

(b) To use the quadratic formula with the last equation above, $a = 1$, $b = -10$, and $c = 20$. This gives the following:

$$\begin{aligned} x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(20)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100-80}}{2} \\ &= \frac{10 \pm \sqrt{20}}{2} \\ &= \frac{10 \pm \sqrt{4}\sqrt{5}}{2} \\ &= \frac{10 \pm 2\sqrt{5}}{2} \\ &= 5 \pm \sqrt{5} \end{aligned}$$

So the numbers are $5 + \sqrt{5}$ and $5 - \sqrt{5}$.

(c) Adding the numbers gives $5 + \sqrt{5} + 5 - \sqrt{5}$, which is equal to 10. Multiplying gives the following:

$$\begin{aligned} (5+\sqrt{5})(5-\sqrt{5}) &= 5^2 - 5(\sqrt{5}) + (\sqrt{5})(5) - (\sqrt{5})^2 \\ &= 25 + 0 - 5 \\ &= 20 \end{aligned}$$

3. (a) Using a similar strategy as above, the numbers would be solutions to the equation $x^2 - 10x + 30$. Using the quadratic formula gives the following:

$$\begin{aligned} x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(30)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 120}}{2} \\ &= \frac{10 \pm \sqrt{-20}}{2} \end{aligned}$$

(b) Since -20 has no square root (among the real numbers), this has no solution.

(c) If there were a number for $\sqrt{-20}$, the problem would have had a solution, so one possible answer is $\sqrt{-20}$.

(d) The answer depends on the number invented. Using $\sqrt{-20}$, the numbers would be $\frac{10+\sqrt{-20}}{2}$ and $\frac{10-\sqrt{-20}}{2}$, or equivalently, $5 + \frac{\sqrt{-20}}{2}$ and $5 - \frac{\sqrt{-20}}{2}$.

(e) Assuming addition and multiplication works the same as with other radicals, the example answers in part (b) give the following. Notice that adding is simpler using one form for the numbers, but multiplying is simpler using the other form.

$$\begin{aligned} 5 + \frac{\sqrt{-20}}{2} + 5 - \frac{\sqrt{-20}}{2} &= 5 + 5 + (\frac{\sqrt{-20}}{2} - \frac{\sqrt{-20}}{2}) \\ &= 10 \end{aligned}$$

Teacher's Note: Students will probably assume the operations work as usual without mentioning it, but the assumption is important to discuss. When extending any mathematical concept, one usually wants to keep existing concepts simple and uniform. That means a goal for using these invented numbers is that operations will, in fact, continue to work as usual. However, as is shown in section 2, in this case the assumption causes a problem that must be fixed.

$$\begin{aligned} (\frac{10 + \sqrt{-20}}{2})(\frac{10 - \sqrt{-20}}{2}) &= \frac{(10 + \sqrt{-20})(10 - \sqrt{-20})}{4} \\ &= \frac{10^2 - 10\sqrt{-20} + 10\sqrt{-20} - (\sqrt{-20})^2}{4} \\ &= \frac{100 - (\sqrt{-20})^2}{4} \\ &= 25 - \frac{(\sqrt{-20})^2}{4} \end{aligned}$$

This final expression doesn't look like 30, but the square root of a number n is a number that, when squared, gives n. So $(\sqrt{-20})^2$ should be equal to -20. Then the last expression is $25 - \frac{-20}{4}$, which is $25 - (-5)$, or 30.

2 Getting precise

1. (a) $\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i$

(b) The square of a square root of a number is the number itself. For example, $(\sqrt{5})^2 = 5$. Since $i = \sqrt{-1}$, $i^2 = -1$.

(c) $i^3 = i(i^2) = i(-1) = -i$

(d) $i^4 = i(i^3) = i(-i) = -i^2 = -(-1) = 1$, or alternatively, $i^4 = (i^2)^2 = (-1)^2 = 1$.

2. (a) $3i + 2i = (3+2)i = 5i$

(b) $3i \times 2i = (3 \times 2)(i^2) = 6(-1) = -6$

(c) $3 + 2i$ can't be combined.

(d) $3 \times 2i = (3 \times 2)i = 6i$

Teacher's Note: The text before problem 2 mentions division, although no division is done in this problem sequence. See the problem sequence "*i* divide."

3. The goal is to write -3 and $4i$ in the form $a + bi$. The expression $a + bi$ has a term that's a real number, and a term that includes i. (The second term is called the *imaginary* part of the complex number.) Since -3 is a real number, we have $-3 = -3 + bi$. For this to be true, $bi = 0$. That means $b = 0$, so $-3 = -3 + 0i$. Similarly, $4i = a + 4i$, so $a = 0$ and $4i = 0 + 4i$.

4. (a) $5 + 7\sqrt{-16} = 5 + 7i\sqrt{16} = 5 + 28i$

(b) $\sqrt{-16} = i\sqrt{16} = 4i = 0 + 4i$.

(c) Using the idea in the previous problem, $67 = 67 + 0i$.

(d) $\sqrt{16} = 4 = 4 + 0i$

5. (a) To expand binomials, use the distributive property multiple times:

$$\begin{aligned}(x+1)(x-6) &= x(x-6) + 1(x-6)\\ &= x^2 - 6x + x - 6\\ &= x^2 - 5x - 6\end{aligned}$$

(b) Complete this using the same method as for binomials, with the added step of simplifying $(\sqrt{5})^2$.

$$\begin{aligned}(3+\sqrt{5})(2-\sqrt{5}) &= 3(2-\sqrt{5}) + \sqrt{5}(2-\sqrt{5})\\ &= 6 - 3\sqrt{5} + 2\sqrt{5} - (\sqrt{5})^2\\ &= 6 - \sqrt{5} - 5\\ &= 1 - \sqrt{5}\end{aligned}$$

(c) Again, use the same method, then simplify i^2.

$$\begin{aligned}(1+7i)(2-2i) &= 1(2-2i)+7i(2-2i)\\ &= 2-2i+14i-14i^2\\ &= 2-12i-14(-1)\\ &= 2-12i+14\\ &= 16-12i\end{aligned}$$

6. (a) $\sqrt{-9}\sqrt{-4}=(3i)(2i)=6i^2=6(-1)=-6$

(b) $(3+2i)+(9-i)=(3+9)+(2-1)i=12+i$

(c) $\begin{aligned}(3+2i)(9-i) &= 27-3i+18i-2i^2\\ &= 27+15i-2(-1)\\ &= 27+15i+2\\ &= 29+15i\end{aligned}$

(d) $\begin{aligned}(7+2i)(3-4i) &= 21-28i+6i-8i^2\\ &= 21-22i+8\\ &= 29-22i\end{aligned}$

(e) $\begin{aligned}(5+2i)+(5-2i) &= (5+5)+(2-2)i\\ &= 10 \text{ (or equivalently, } 10+0i)\end{aligned}$

(f) $\begin{aligned}(5+2i)(5-2i) &= 25+10i-10i-4i^2\\ &= 25+4\\ &= 29 \text{ (or equivalently, } 29+0i)\end{aligned}$

(g) $\begin{aligned}(a+bi)+(a-bi) &= (a+a)+(b-b)i\\ &= 2a \text{ (or equivalently, } 2a+0i)\end{aligned}$

(h) $\begin{aligned}(a+bi)(a-bi) &= a^2-abi+abi-b^2i^2\\ &= a^2+b^2 \text{ (or } a^2+b^2+0i)\end{aligned}$

(i) $\begin{aligned}(a-bi)+(c-di) &= a+c-bi-di\\ &= a+c-(b+d)i\end{aligned}$

(j) $\begin{aligned}(a-bi)(c-di) &= ac-adi-bci+bdi^2\\ &= ac-bd-(ad+bc)i\end{aligned}$

7. (a) If the numbers are x and y, $x+y=10$ and $xy=30$. Solving the first for y gives $y=10-x$. Substituting for y in the second then gives the following:

$$\begin{aligned}x(10-x) &= 30\\ 10x-x^2 &= 30\\ -x^2+10x-30 &= 0\\ x^2-10x+30 &= 0\end{aligned}$$

Using the quadratic formula gives

$$\begin{aligned}
x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(30)}}{2(1)} \\
&= \frac{10 \pm \sqrt{100 - 120}}{2} \\
&= \frac{10 \pm \sqrt{-20}}{2} \\
&= \frac{10 \pm i\sqrt{20}}{2} \\
&= \frac{10 \pm i\sqrt{4}\sqrt{5}}{2} \\
&= \frac{10 \pm 2i\sqrt{5}}{2} \\
&= \frac{2(5 \pm i\sqrt{5})}{2} \\
&= \frac{2}{2}(5 \pm i\sqrt{5}) \\
&= 5 \pm i\sqrt{5}
\end{aligned}$$

So the numbers are $5 + i\sqrt{5}$ and $5 - i\sqrt{5}$.

(b) Using a similar strategy as in part (a) gives $x^2 - sx + p = 0$. Using the quadratic formula gives

$$\begin{aligned}
x &= \frac{-(-s) \pm \sqrt{(-s)^2 - 4(1)(p)}}{2(1)} \\
&= \frac{s \pm \sqrt{s^2 - 4p}}{2}
\end{aligned}$$

So the numbers are $\frac{s+\sqrt{s^2-4p}}{2}$ and $\frac{s-\sqrt{s^2-4p}}{2}$.

(c) The numbers are real when the radicand, $s^2 - 4p$, is non-negative ($s^2 - 4p \geq 0$, or equivalently, $s^2 \geq 4p$).

(d) Since real numbers are also complex, the numbers are complex for all values of s and p.