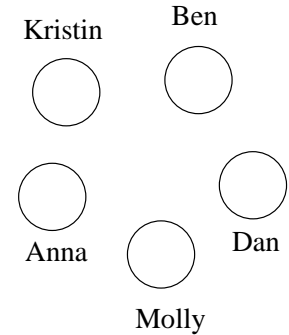


Arrangements in a circle Problem

1. Three girls and ten boys are sitting in a circle. All the girls are sitting together, and all the boys are sitting together. How many pairs of neighboring boys are there? How many pairs of neighboring girls?
2. How many neighboring pairs are the same gender?
3. How many neighboring pairs are of different genders?
4. If a new girl comes and sits in this circle, how would the number of girl-girl and boy-boy pairs change?
5. Can we rearrange these students so that the number of same-gender pairs decreases? If so, how? Can we make this number increase? If so, how?
6. How big can this number get? How small?
7. Now the circle contains ten girls and seven boys. At most, how many same-gender neighbor pairs can there be? What is the smallest number of same-gender pairs possible?
8. Fifteen boys and some girls are in the circle. There are ten neighboring pairs of boys. How many girls could there be?
9. In a certain circle, six pairs of girls are neighbors, and 4 pairs of boys. How many kids could there be in the circle?
10. If ten girls and 15 boys are in the circle and there are 12 pairs of neighbor boys, how many pairs of neighbor girls could there be?
11. **Generalize:** Write formulas for the largest and smallest number of pairs of the same gender for *any* number of boys and girls in a circle. How did you get these formulas?
12. **Modify the problem:** Answer questions 1–11 if the students are sitting in a row, instead of in a circle. Instead of solving all these new problems from scratch, try to find a way to modify the solutions you already have.
13. **Generalize further:** Forget about gender and think about languages. What could be the largest and smallest numbers of neighboring pairs speaking the same language, if there are x students speaking English, y students speaking French, and z students speaking Spanish?

You may want to use checkers or counters of two different colors to represent this situation

A pair of students is different from another pair if it does not contain the same two students.



For example, Ben–Dan and Dan–Ben is the same pair; and Dan–Ben and Kristin–Ben are different pairs.

Is the answer different if the three languages are Portuguese, Italian, and Russian?

Hints

Hint to problem 1. Draw a picture.

Hint to problem 4. Consider the various places she can choose to sit.

Hint to problem 7. What would be the arrangements with the smallest and the biggest numbers of pairs of the same gender?

Hint to problem 8.

- Imagine (and draw) a pair as connected with a link. How many of these links are there altogether?
- How many of these links **have to be** boy-boy links?
- How many of these links **have to be** boy-girl links?
- What is the minimum number of girls that have to be in this circle?
- Do you have any restrictions for a number of girl-girl links?
- Do you have any restrictions for a number of girls?

Hint to problem 9. What is the least number of students? How many more students could be there and how could they be arranged?

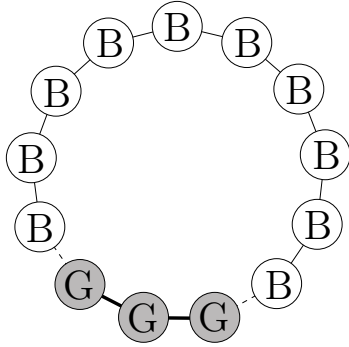
Hint to problem 12. If you consider a pair as connected with a link, as in a chain, what changes if you switch from a circle to a line? If one of the links is broken, what type of link could it be (same gender, different gender, anything else?)

Answers

1. 9,2.
2. 11
3. 2
4. Either one girl-girl link is added, or one boy-boy link is replaced by two boy-girl links.
5. The number of pairs of same gender decreases when the boys and girls are mixed. The original arrangement creates the greatest number of neighboring pairs of the same gender.
6. 11,7
7. Any odd number from 3 to 15.
8. Any number greater than 5.
9. Any even number greater than 12.
10. Seven.
11. The largest is $g + b - 2$, the smallest is $b - g$. That means that it is $b - g$, if $b > g$; or it is $g - b$, if $g > b$.
12. **12(1)**. Same as 1: 9,2
12(2). Same as 2: 11
12(3). One
12(4). Either one girl-girl link is added, or one boy-girl link is added, or one boy-boy link is replaced by two boy-girl links.
12(5). Same as 5
12(6). 11,6
12(7). 2 - 15
12(8). Any number greater than 4
12(9). Any number greater than 12
12(10). 6, 7, or 8.
12(11). $b + g - 2$, $|b - g| - 1$
13. Smallest: 0, largest: $x + y + z - 3$.

Solutions

- Imagine (or draw) boys and girls in a circle connected by links as in a circular chain. You can use different colors for boy-boy, girl-girl, and boy-girl links. There are 9 boy-boy links. And there are 2 girl-girl links.

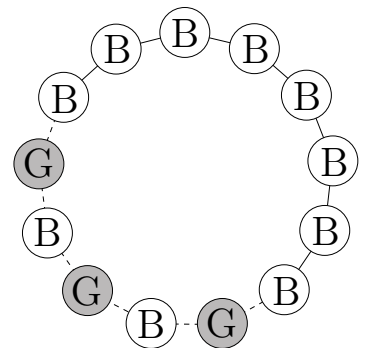


In this picture, same-gender pairs are connected with solid links and different gender pairs are connected with dashed links. We will use the same conventions in all of our illustrations.

- There are 11 links connecting students of the same gender (9 boy-boy + 2 girl-girl links).
- There are 2 girl-boy links.
- If an extra person comes, one more link is added to the circle. If a girl sits among girls or between a girl and a boy, this extra link is a girl-girl link. If she sits among boys, one boy-boy link is replaced by two boy-girl links.
- A general rule: The total number of pairs in a circle must stay the same, so a decrease in same-gender pairs causes an increase in pairs of different gender, and vice versa. If no girls sit together, the number of same-gender pairs decreases, so the number of different-gender pairs must increase.

The original arrangement creates the greatest number of neighboring pairs of the same gender.

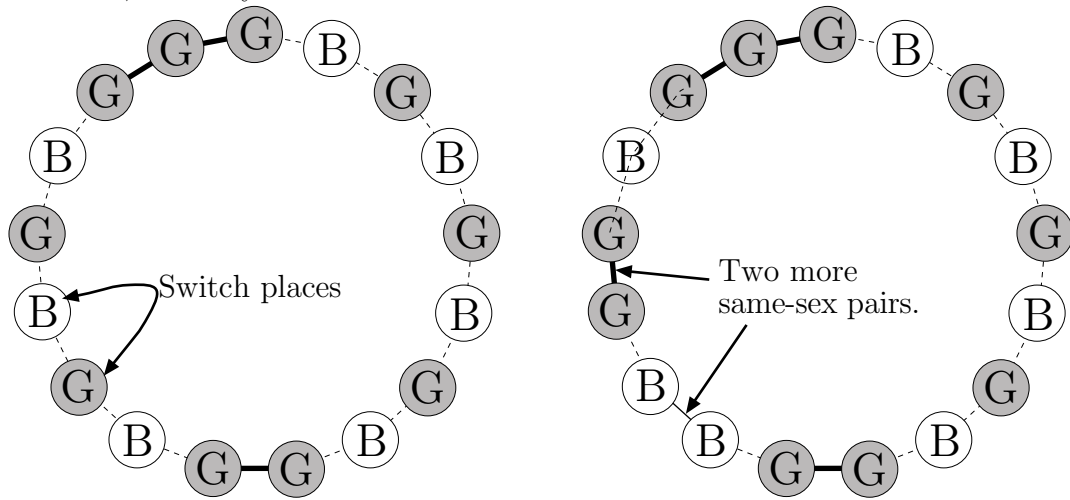
- There can be, at most, 11 same-gender pairs. (Use the methods from problem 2.) This number is smallest when none of the girls are together. Then there are 7 pairs of neighboring boys and no pairs of neighboring girls.
- The largest number of same-gender pairs occurs when all girls sit together: 9 pairs of neighboring girls and 6 pairs of neighboring boys, with a total of 15 same-gender pairs. If we seat girls between all the boys, there will be no pairs of neighboring boys and three pairs of neighboring girls. The picture on the left, below, shows one such arrange-



ment, with a total of three pairs of same-sex neighbors.

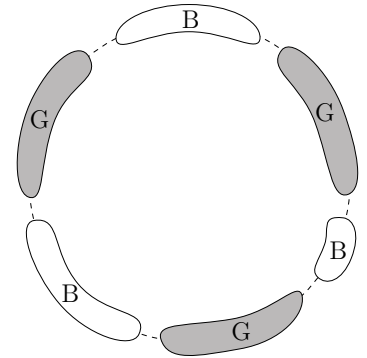
In such an arrangement, if we switch places of one boy and one girl, we increase a number of same-sex pairs by two. So we can get every odd number of same-sex pairs between 3 and 15, but only the odd numbers.

This will work even if the boy and girl aren't neighbors! (Draw a picture to confirm that.)



8. If there are fifteen boys in a circle, but only ten pairs of neighboring boys, then there must be at least five girls separating them. (Think about the girls as breaks in a continuous circle of boys). However, you can put any number of girls next to each of these five girls without changing the number of neighboring boys. Therefore there could be five or more girls in this circle.

9. If there are to be six pairs of girls, and four pairs of boys, then there must be at least seven girls and five boys in the circle, with the total of 12 children. If you add a boy-girl pair, you don't change the number of same gender pairs. (Check it on your drawing.) Therefore you can have any number of boy-girl pairs in this circle with 6 girl-girl and 4 boy-boy pairs. You can have 12 or more students, but only even numbers: 12, 14, 16, etc.



10. If there are fifteen boys, but only twelve neighboring pairs of boys, it means that there are three clusters of boys separated by girls, and therefore, three clusters of girls. This means that there are six boy-girl pairs. There are total of 25 links in this circle: 12 boy-boy pairs, and 6 boy-girl pairs. Therefore, there are exactly 7 girl-girl pairs in this circle.

- 11.** The largest number of same gender pairs occurs when all boys sit together and all girls sit together. g girls form $(g - 1)$ girl-girl links, and b boys form $(b - 1)$ boy-boy links, for a total of $g + b - 2$ same gender pairs. If there are more boys than girls ($b > g$), then the smallest number of same gender pairs occurs when each girl has a boy on both sides. (Draw a picture and verify that that is true.) Since the total number of links is $b + g$, and the number of girl-boy links on two sides of each girl is $2g$, then the number of boy-boy links will be $(b + g) - 2g = b - g$. If there are more girls than boys, the number will be $g - b$.
- 12.** Imagine that you broke one link in a circular chain, then you have a linear chain. Therefore, for each situation in a circle, there will be a similar situation in a row with one broken link. The answers will depend on whether you've broken a same gender or different gender link.
- 12 (1-3).** Since all boys are still sitting together and all girls are sitting together, it means that the boy-girl link was broken. Therefore, there are still 9 boy-boy and 2 girl-girl pairs, but only 1 boy-girl pair.
- 12(4).** If a girl sits next to or among girls, then 1 girl-girl link will be added. If she sits between two boys, 2 girl-boy links will be added and one boy-boy link will disappear. If she sits on the end of boys' side, one boy-girl link will be added.
- 12(6).** The largest possible number for the number of same gender pairs is still 11, and the smallest is 6, because you can arrange boys on both ends of the row, as if the boy-boy link was broken.
- 12 (7).** From 2 (if the boy-boy link is broken) to 12 (if the boy-girl link is broken) same gender pairs.
- 12(8).** If the boy-boy link is broken, there must be at least four girls to separate boys. So there could be 4 or more girls in this row.
- 12(9).** Now you can add even or odd number of boys and girls arranged in boy-girl pairs. So you can have 12 or more children in this row.
- 12.(10)** In order for fifteen boys in a line to have only twelve neighboring pairs, they must be divided into three clusters. Possible arrangements of three clusters of boys will be:

$$BGBGB$$

when a boy-boy link is broken, or

$$GBGBGB$$

when a boy-girl link is broken, or

$$GBGBGBG$$

when a girl-girl link is broken.

Therefore, there are either 6, 7, or 8 pairs of neighboring girls.

12(11). Similarly to problem 6, the largest number of same gender pairs occurs when all boys sit together and all girls sit together. The number is $g + b - 2$. The arrangements are GB or BG, with a boy-girl link broken. The smallest number of same gender pairs occurs when boys and girls are mixed, and a same gender link is broken. So if there are more boys than girls, the number is $b - g - 1$. If there are more girls than boys, the number is $g - b - 1$. In general, it is $|b - g| - 1$

- 13.** The smallest number of neighboring pairs speaking the same language can be 0. In this case the following constraints on the x , y , and z should hold: $x \leq y + z$ and $y \leq x + z$ and $z \leq x + y$. The largest number of neighboring pairs speaking the same language would be $x + y + z - 3$, if all students speaking the same language sit together.