Divisibility

In these problems, all numbers mentioned are **integers**.

Integer numbers:
..., -3, -2, -1, 0, 1, 2, 3, ...

If $km = n$, where k, m, and n are all integer numbers, then n is a **multiple** of both m and k, and numbers m and k are **factors** of n.

1. True or false?

(a) If a is a multiple of b, and b is a multiple of c, then a is a multiple of c.

(b) If a is a multiple of c, and b is a multiple of c, then $a+b$ and $a-b$ are also multiples of c.

(c) If a is *not* a multiple of c, and b is *not* a multiple of c, then $a+b$ is *not* a multiple of c.

(d) If a is *not* a multiple of c, and b is a multiple of c, then $a+b$ is *not* a multiple of c.

(e) If ab is a multiple of c, then a is a multiple of c or b is a multiple of c.

(f) If a is a multiple of 15 and b is a multiple of 24, then ab is a multiple of 360.

(g) If a is a multiple of 15 and of 24, then a is a multiple of 360.

2. Prove that for any integer n, the product $n(n+1)$ is even.

3. Prove that the product of any three consecutive numbers is a multiple of 6.

4. Without calculating the sum, prove that the sum $73+74+75+76+77+78+79$ is a multiple of 76 and of 7.

5. Suppose you have natural numbers a, b, c, and d, where $ab = cd$, and a is a multiple of c. Prove that d is a multiple of b.

6. Prove that if $a+2$ and $12-b$ are multiples of 10, then $a+b$ is a multiple of 10.

7. How many numbers between 1 and 80 have odd number of factors (including 1 and itself)?

8. How many numbers from 1 to 80 are

(a) multiples of 2;

(b) multiples of 3;

(c) multiples of 2 and 3;

(d) multiples of 2 but not 3;

(e) neither multiples of 2 nor multiples of 3?

Hints

Hint to problem 1. To prove that a statement is false it's enough to give one example for which the statement is false (it is called **a counterexample**). However, to prove that a statement is true, it is not sufficient to provide one or more examples: one has to show that it is true *always*.

Hint to problem 1d. Use results of 1b.

Hint to problem 3. To prove that a number is a multiple of 6, prove that it is a multiple of 2 and 3.

Hint to problems 5 and 6. Use the definition of a multiple.

Hint to problem 8d and 8e. Use Venn diagram.

Answers

1. (a) True.
(b) True.
(c) False.
(d) True.
(e) False.
(f) True.
(g) False.

2. See solutions.

3. See solutions.

4. See solutions.

5. See solutions.

6. See solutions.

7. 8 numbers

8. (a) 40
(b) 26
(c) 13
(d) 27
(e) 27

Solutions

1. (a) This is true. Since a is a multiple of b, by definition $a = kb$, where k is an integer. But b in its turn is a multiple of c, and so $b = mc$, where m is an integer. Substituting this expression for b gives $a = k(mc) = (km)c$, and so a is a multiple of c.

(b) This is true. Since a is a multiple of c, by definition $a = kc$, where k is an integer. Since b is a multiple of c, $b = mc$, where m is an integer. Then $a + b = kc + mc = (k + m)c$ and $a - b = kc - mc = (k - m)c$. Both $k + m$ and $k - m$ are integers, so $a + b$ and $a - b$ are also multiples of c.

(c) This is not true. Example: neither 5 nor 7 are multiples of 3, but their sum is.

(d) This is true. Suppose there are such a, b, and c that b and $a + b$ are both multiples of c while a is not a multiple of c. Since a can be expressed as the difference between $a + b$ and b, it follows from 1b that a also has to be a multiple of c.

(e) This is not true. Example: neither 2 nor 3 is a multiple of 6, but their product is.

(f) True. Since a is a multiple of 15, it can be expressed as $15k$. Since b is a multiple of 24, it can be expressed as $24m$. Then $a \cdot b = 15k \times 24m = 15 \times 24km = 360km$, and ab is a multiple of 360.

(g) Not true. Example: 120 is a multiple of both 15 and 24 but not of their product 360.

2. Numbers n and $n + 1$ are consecutive, and one of the two consecutive numbers is always even. From 1a follows that all the multiples of that even number, and $n(n+1)$ among them, are also even.

3. A number is a multiple of 6 if it is even and a multiple of 3. It follows from problem 2 that the product of three consecutive numbers is even. Since one of any three consecutive numbers is a multiple of 3, it follows from 1a that the product of three consecutive numbers is a multiple of 3.

4. "Move" 1 from 77 to 75 (the value of the sum will not change):

$$73+74+75+76+77+78+79 = 73+74+76+76+76+78+79.$$

In the same manner, move 2 from 78 to 74 and 3 from 79 to 73:

$$73+74+75+76+77+78+79 = 73+74+76+76+76+78+79 =$$

$$= 76 + 76 + 76 + 76 + 76 + 76 + 76 = 7 \times 76.$$

By definition 7×76 is a multiple of both 7 and 76.

5. Since a is a multiple of c, $a = kc$. Substituting a in $a \cdot b = c \cdot d$ by kc and dividing both parts by c gives the following:

$$\begin{aligned} ab &= cd \\ kcb &= cd \\ kb &= d \end{aligned}$$

By definition d is a multiple of b.

6. It is given that $a + 2 = 10k$ and $12 - b = 10m$. So, $a = 10k - 2$ and $b = 12 - 10m$. Then $a + b = 10k - 2 + 12 - 10m = 10k + 10 - 10m = 10(k + 1 - m)$. So, by definition, $a + b$ is a multiple of 10.

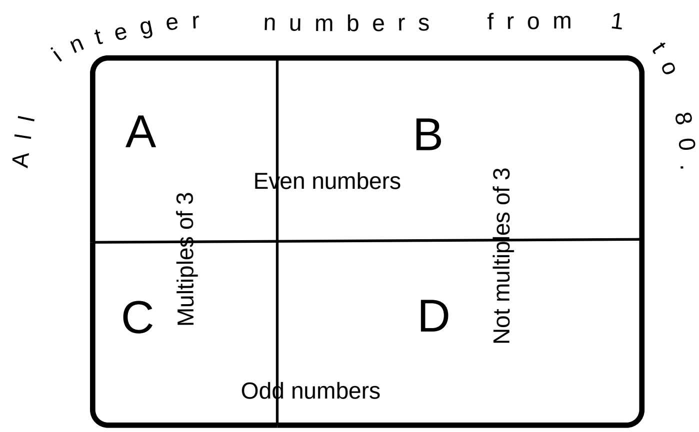
7. Only perfect squares have odd number of factors, and here's why: if k is a factor of n, then $n \div k$ is also a factor of n. So, it is always possible to group factors of n in pairs, the product of each pair being equal to n, unless k and $n \div k$ are equal. In this case, $n = k^2$, and the total number of factors of number n is odd.
There are 8 perfect squares among numbers from 1 to 80.

8. (a) Every other number is even; there are 80 numbers from 1 to 80, half of them being even. Answer: 40.

(b) Every third number is a multiple of 3. However, $80 \div 3$ is not a whole number, but $81 \div 3$ is. Since $81 \div 3 = 27$, there are 27 multiples of 3 from 1 to 81. All of these multiples except 81 are from 1 to 80. Answer: 26.

(c) Multiples of 3 and 2 are multiples of 6. The biggest multiple of 6 in the range from 1 to 80 is 78. Since every 6th number is a multiple of 6, there are 13 ($78 \div 6 = 13$) multiples of 6 from 1 to 78.

(d) This can be done by using Venn diagram.



On this diagram, region A represents all numbers from 1 to 80 that are multiples of 2 and 3. (There are 13 of them, as was found in 8c.) Region B represents even numbers that are not multiples of 3, the number of which we are to find. All even numbers are in A or B, and there are 40 of them (from problem 8a). Therefore, there are 27 even numbers that are not multiples of $3(40 - 13 = 27)$.

(e) From the same Venn diagram, the number of those numbers that are not multiples of 2 or 3 is $80-(A+B+C)$. There are 26 numbers in $A+C$ (from problem 8b) and 27 numbers in B (from problem 8d). Therefore, there are 27 numbers in D.

