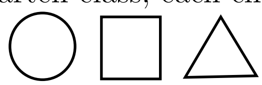
Are they different or the same? Problem

This sequence is adapted from *Mathematical Methods in High School*, EDC.

1 How many three digit numbers can you make using only the digits 1 and 2? Some of these numbers are 111, 112, 221. (Of course, in each number you may use a digit more than once.)

2 In a kindergarten class, each child is given a picture:



and asked to color each of 3 different shapes in it either green or red. If all of the kids do their job correctly, at most how many different pictures can there be?

3 A pizzeria has three choices of toppings: onions, mushrooms, and pepperoni. On the order form, you check off the toppings you want, if any. How many different pizzas are possible?

onions	mushrooms	pepperoni

You may choose not to check off any topping — this would be a plain pizza.

4 A coin is tossed three times. One of the possible outcomes is *tail-head-head*, another one is *head-tail-head*. How many possible outcomes are there?

5 What (besides the answer) is the same in problems 1–4?

Problems 1–4 are all *isomorphic*. **Isomorphic** is from Greek *iso*- “same” and *morphus*- “shape.” This word is used in different parts of mathematics to refer to structures that are essentially the same. Isomorphic problems have the same mathematical structure and can be solved using the same method, or the same formula or equation.

6 Let’s imagine now, that problem 1 is slightly changed:

“How many *four* digit numbers can you make using only the digits 1 and 2?”

(a) Solve this new problem.

(b) Make up a problem isomorphic to this one.

Hints

Working on these problems, look for similarities in their structure.

Answers

1. 8 numbers.

2. 8 pictures.

3. 8 pizzas.

4. 8 outcomes.

5. Problems 1–4 have the same mathematical structure. One of the ways to solve each of problems 2–4 is to show that it is “the same” as problem 1 (see the last suggested solution for problems 2–4).

6. (a) 16 numbers.

(b) Examples:

- The kindergarten teacher asks the children to color each of 4 different shapes green or red. If all of the kids do their job correctly, how many different pictures can there be?
- A pizzeria has four choices of toppings: onions, mushrooms, pepperoni, and olives. How many different pizzas are possible?
- A coin is tossed four times. How many possible outcomes are there?

Solutions

1. Here are two of many possible ways to solve this problem:

- Carefully list all these numbers in some order (for example, increasing) and then count them.
- There are two choices for the first digit of a number; *in each case*, there are two choices for the second digit, so there is a total of $2 \cdot 2 = 4$ possibilities for the first two digits. For each of these 4 ways to start, the last digit can be either 1 or 2, for a total of $4 \cdot 2 = 8$ numbers.

2. Here are three (of many) ways to solve this problem:

- List all the ways to color the picture and then count.
- There are two choices for the color of the circle. *For each choice*, there are two choices for the square, so there are $2 \cdot 2 = 4$ ways to color the two shapes. For each of these 4 starts, there are two ways to color the triangle. So there are $4 \cdot 2 = 8$ ways to color the picture.
- We can code each way of coloring the picture. If a shape is colored red, we'll write "1," and if it is colored green, we'll write "2," For each way to color the picture we'll have a different three-digit number made of ones and twos. So, the number of different pictures is the same as the number of different three digit numbers made of ones and twos. The answer to this problem is the same as the answer to problem 1.

3. Some of the ways to solve this problem:

- List all different pizzas and count them.
- Each topping can be either checked off or not, so there are $2 \cdot 2 \cdot 2 = 8$ different pizzas.
- We can code each pizza by writing 1 in the order form if we want a particular topping and 2 if we don't. This way each pizza is coded by a three digit number made of ones and twos; so the answer to this problem is the same as the answer to problem 1.

If you think of checking off a topping as "giving it the green light," and rejecting a topping as "giving it the red light," then this is the same as the shape-coloring problem.

4. Some of the ways to solve this problem:

- List all different outcomes and count them.
- Each toss can be either head or tail, so there are $2 \cdot 2 \cdot 2 = 8$ different outcomes.
- We can code each outcome by writing 1 for a head and 2 for a tail. This way each outcome is coded by a three digit number made of ones and twos; so the answer to this problem is the same as the answer to problem 1.

Or we can paint the heads red and the tails green. Then each outcome is a way of coloring three objects, like problem 2.

5. Problems 1–4 have the same mathematical structure. One of the ways to solve each of the problems 2–4 is to show that it is "the same" as problem 1 (see the last suggested solution for problems 2–4).

6. (a) One way to generate all four digit numbers made of ones and twos is to write 1 or 2 at the end of each three digit number made of ones and twos. Therefore, there are twice as many four digit numbers made of ones and twos as there are three digit numbers.

Of all possible four digit numbers made only of ones and twos, how many have the last digit 1? We have to look at the other three digits, so that is just problem 1! And what if the last digit is 2? Same thing!

(b) Examples:

- The kindergarten teacher asks the children to color each of 4 different shapes green or red. If all of the kids do their job correctly, how many different pictures can there be?
- A pizzeria has four choices of toppings: onions, mushrooms, pepperoni, and olives. How many different pizzas are possible?
- A coin is tossed four times. How many possible outcomes are there?

