

An even and an odd Problem

Inspired by *Mathematical Circles* by Dmitry Fomin, Sergey Genkin, and Ilia Itenberg

While doing these problems, keep in mind the title of this set of problems. Don't forget to explain the answer!

1. Can a 5×5 board be cut into 1×2 dominoes?

2. Eleven gears are arranged in a chain as shown here. Can the gears rotate?

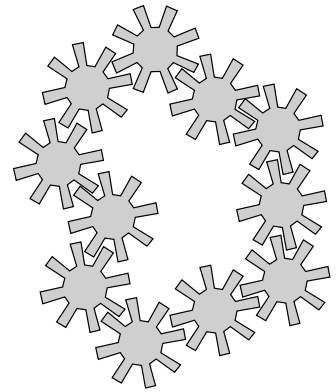
3. Thirteen line segments are connected end-to-end, so that they form a path. Is it possible that each of these segments crosses exactly one of the other segments?

4. Can there be a magic square made of the first 36 primes?

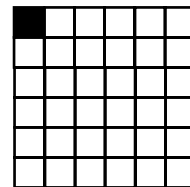
5. In a 6×6 chart all but one corner black square are painted white. You are allowed to repaint any column or any row in the chart (i.e., you can select any row or column and change color of all squares within that line). Is it possible to attain an entirely white chart by using only the permitted operations?

6. John and Pete have three pieces of paper. Each of the boys picks one piece, tears it up, and puts the smaller pieces back. John only tears a piece of paper into 3 smaller pieces while Pete only tears a piece of paper into 5 smaller pieces. After a few minutes can there be exactly 100 pieces of paper?

If the answer to the problem is "yes" (the checkboard *can* be cut into dominoes), it's enough to just show a method of cutting it. But, if the answer is "no" (it is impossible to cut the checkboard into dominoes), you should show not only that *one* method does not work, but that no method *whatsoever* would work.



A magic square here means a 6×6 chart of numbers, so that the sum of numbers along any column, row, or diagonal is the same.



Hints

Hint to problem 1. Assume one gear rotates clockwise and see how the other ones behave.

Hint to problem 3. Consider pairs of intersecting segments.

Hint to problem 4. Must the sum in each row (and column) be even or odd?

Hint to problem 5. How does the number of white squares change after applying an operation?

Hint to problem 6. Look at the number of pieces after each tearing.

Answers

1. No.
2. No.
3. No.
4. No.
5. No.
6. No.

Using parity (evenness or oddness of a number) is very helpful for *disproving* things, and not that helpful for proving them.

Solutions

1. No, because there is an odd number of squares, and each domino takes 2 squares. One square will be left unused.
2. Assume one gear rotates clockwise. Then both of its neighbors must rotate counter-clockwise, the third gears (in both directions) must rotate clockwise again, and so on: the “odd” gears must rotate clockwise, while the “even” gears must rotate counter-clockwise. But then the first and the eleventh gears must rotate the same direction, and this is impossible, because they are neighbors. We have proven that these eleven gears cannot rotate.
3. If such a path were possible, its line segments could be grouped into pairs of intersecting segments. But then the number of segments would have to be even. Therefore, such a path with any odd number of segments is impossible.
4. Among the first 36 primes, one is even (number 2) and all the rest are odd. The sum in the row which contains 2 must be odd (because it is the sum of five odd numbers and one even number), while the sum in other rows must be even (six odd numbers). Therefore, the sum cannot be the same for all rows.
5. When color of each square in a row (or a column) is changed, the parity (evenness or oddness) of black squares in that row (or a column) does not change. That’s why there will always be an odd number of black squares in the chart, and the chart will never be entirely white.
6. When John tears a piece of paper, the number of pieces increases by 2, and when Pete tears a piece of paper, the number of pieces increases by 4. The parity of the number of pieces does not change through the whole process. Therefore, since originally there were three pieces, the number of pieces will always remain odd and can never be equal to 100.

For example, if there was one black square in a row, after the color of that row is colored, there will be 5 black squares in it — still an odd number of black squares!