An even and an odd — 2
Problem

Inspired by *Mathematical Circles* by Dmitry Fomin, Sergey Genkin, and Ilia Itenberg

1. The product of 30 integers is equal to 1. Can their sum be equal to zero?

2. Can an ordinary 8×8 chessboard be covered with 1×2 dominoes so that only two corner squares on the same diagonal remain uncovered?

3. A box is filled with 75 white beads and 150 black ones. There is a pile of black beads near the box. Remove two beads from the box. If one is black, put back the other (white or black). If both are white, put in a black one from your pile. Each time one repeats this process, there will be one less bead in the box. What will be the color of the final bead left in the box?

4. Four integers a, b, c, and d produce 6 pairwise sums 2, 4, 9, 9, 14, 16. Is that possible?

If a, b, c, and d are not necessarily integers then what are their values?

5. A snail crawls along with constant velocity, turning through a right angle every 15 minutes. Show that the snail can return to its starting point only after a whole number of hours.

6. There are 100 soldiers, and every evening three of them are on duty. Is it possible that after a certain number of days each soldier was on duty with every other soldier exactly once?

7. All natural numbers from 1 to 101 are written in a row. Can the signs "+" and "−" be placed between them so that the value of the resulting expression is 0?

8. Of 101 coins, 50 are counterfeit, and they differ from the genuine coins in weight by 1 gram. Peter has a scale in the form of a balance which shows the difference in weight between the objects placed on each pan. He chooses one coin, and wants to find out whether it is counterfeit. Can he do this in one weighing?

9. Can a convex nonagon (a polygon with 9 sides) be cut into parallelograms?

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Hints

Hint to problem 2. The two corner squares of a chessboard that are on the same diagonal always have the same color.

Hint to problem 4. How many of a, b, c, and d are even and how many are odd?

Hint to problem 5. Suppose the snail's path has been drawn so that it has vertical and horizontal segments. Show that the number of vertical and horizontal segments of the snail's path is the same.

Hint to problem 6. If yes, how many times would each soldier have been on duty?

Hint to problem 7. If one of the "+" in the expression is changed to "−", how would the value of the whole expression change?

Hint to problem 9. Suppose it is possible to cut a nonagon into parallelograms. Consider one of the sides of the nonagon. There is a parallelogram which has its side on the chosen side of the nonagon. "Jump" from this side to the parallelogram's other side, parallel to it. This side must be shared with another parallelogram. Jump to that other parallelogram's parallel side. Where can you end up?

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Answers

1. No.

2. No.

3. The final bead in the urn will be white.

4. Numbers a, b, c and d cannot be integers.

If a, b, c and d do not have to be integers, they are -1.5, 3.5, 5.5, and 10.5.

5. See solutions.

6. No.

7. No.

8. Yes.

9. No.

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Solutions

1. Every integer must be equal to 1 or -1. Suppose their sum *is* equal to zero. If so, there must be the same number of ones and negative ones, so there must be a total of 15 negative ones. The product of an odd number of negative ones is -1, so the product of these 30 numbers cannot be 1.

2. The chessboard has the same number of white and black squares; the uncovered corner squares of the chessboard are of the same color. Each domino covers one black and one white square of the chessboard; so it is impossible that two squares of the same color remain uncovered.

3. White beads leave the box in pairs, so there's always an odd number of white beads in the box. Therefore, the last bead in the box will be white.

4. Among four integers there can be four, three, two, one, or no odd numbers. In the first and the last cases all six pairwise sums must be even. In the second and the fourth cases there must be three odd and three even pairwise sums. In the third case there must be four odd and two even sums. So, it is impossible that four integers produce two odd and four even sums.

5. If the snail returned "home" after tracing N vertical segments, then the snail has also traced N horizontal segments. Since the snail has returned home, N must be even. For every segment traced in "away from home" direction there must be one traced in "towards home" direction. Since N is even, $2N$ (the total number of segments) is a multiple of 4, and $2N \times 15$ minutes is a whole number of hours.

6. If a given soldier shared duty with every other soldier exactly once, these remaining 99 soldiers could be divided into pairs, because each time there were exactly 3 soldiers on duty. This is impossible since 99 is an odd number.

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7. If there is "+" in front of each of the numbers, the value of the resulting expression is

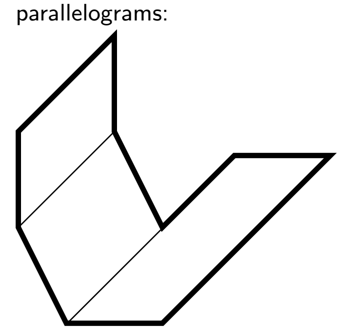
$$1 + 2 + \ldots + 99 + 100 + 101 = \frac{(1 + 101) \cdot 101}{2} \cdot 2 = 5151.$$

If a "+" in front of one of the numbers is changed into "-", the value of the whole expression becomes twice that number smaller. So it becomes an even number smaller, and the parity of the value of the expression remains the same. Therefore, no matter how the signs "+" and "-" are placed, the value of the expression will remain odd and can never be equal to 0.

8. Peter must lay the chosen coin aside, divide the remaining coins into two piles of 50 coins each, and weigh these piles against each other. If the chosen coin is genuine, the difference between the weights of the piles must be even, otherwise it must be odd.

9. Suppose the answer is "yes," and we have succeeded in dividing a nonagon into parallelograms. Choose one side of the nonagon, and consider a parallelogram which has its side on it. The opposite side of this parallelogram is either another side of the nonagon or is also a side of another parallelogram. Consider this new parallelogram and its opposite side, and move in this fashion from the starting side to an opposite side until reaching a nonagon's side. All sides in this chain are parallel, so this final side must be parallel to the side of the nonagon we have started. Therefore, each side of the nonagon must have a parallel one, which is impossible for a convex polygon with an odd number of sides.

It is important for a nonagon to be convex. Example of a concave nonagon divided into parallelograms:



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