Three special triangles
Problem

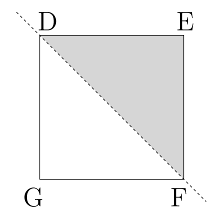
In this problem sequence, you will learn some of the geometry of the three triangles that you are likely to encounter most often: the equilateral triangle, half of it, and the half-square.

Equi means "equal" and *lateral* refers to "sides." An equilateral triangle has sides of equal length.

Measuring their angles and sides

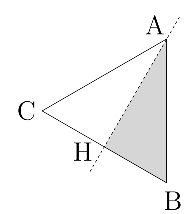
1. The dotted line is a line of symmetry—the gray and white triangles precisely match each other and are each half of square $DEFG$. Tell how that one fact lets you *figure out* the measures of $\angle FDE$, $\angle DEF$, and $\angle EFD$ (even if you did not already know those angles).

Well, you do need one more fact: the measure of the angles in a square!



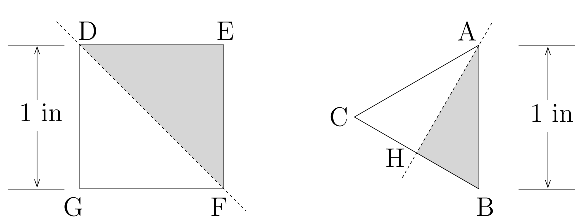
2. Suppose that you don't know the angles in $\triangle ABC$, but you do know that all three angles match. How does that one fact tell you how big those angles must be?

Well, you do need one more fact: that the sum of the angles in any triangle is the same as the sum of the angles in any other triangle.



3. If you know that $\triangle BAH$ precisely matches $\triangle CAH$ (because they're each half of $\triangle ABC$), how does that tell you the measures of $\angle BAH$, $\angle AHB$, and $\angle HBA$?

4. Using the following measurements, how long is side $\overline{HB}$? What fact given earlier lets you figure out that length?



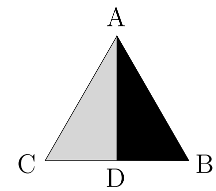
5. Whenever you know the lengths of two sides of a right triangle, the Pythagorean theorem lets you figure out the length of the third side. How long are sides $\overline{AH}$ and $\overline{DF}$?

Getting to know them well

It is worth *remembering* the two numbers you just found, and knowing how they describe the geometry of the triangles. Here are some problems to help you do that.

Equilateral triangles, isosceles right triangles, and 30°-60°-90° triangles turn up quite often in real-life problems and also in school-life tests.

6. $\triangle ABC$ is equilateral, and $\angle ADB$ is a right angle.

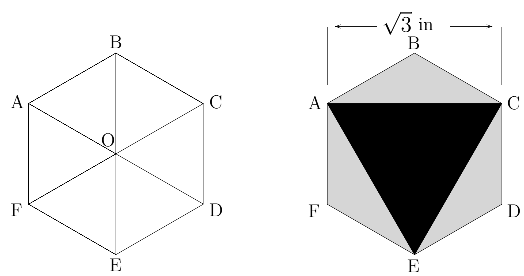


(a) If $DB = 1$, how long are $\overline{AD}$ and $\overline{AB}$?
(b) If $AB = x$, how long are $\overline{AD}$ and $\overline{DB}$?
(c) If $DC = x$, how long are $\overline{AD}$ and $\overline{AC}$?
(d) If $AD = 5$, how long are $\overline{DB}$ and $\overline{AB}$?
(e) For *any* size of $\overline{AB}$, what is the ratio of AD to DB? And what is the ratio of AB to DB?

7. This picture shows a regular hexagon $ABCDEF$ carved up in two different ways. On the left, you see $ABCDEF$ made up of six equilateral triangles. On the right, you see it composed of one large equilateral triangle colored black, and three matching isosceles triangles shaded gray.

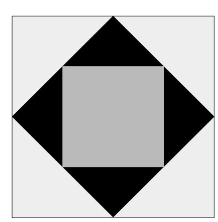
The picture on the right has the annoying habit of sometimes looking like a gray cube with the front corner cut off in a special way, and the cut surface painted black. Other times it looks like the gray hexagon it was intended to be, with a black triangle in it.

Sometimes, the picture on the left insists on becoming a cube, too!



(a) From one of the pictures, you can deduce that each interior angle of the regular hexagon—$\angle ABC$, $\angle BCD$, and so on—has the same measure. What is that measure? How can that be deduced from the picture?
(b) Figure out the angles in $\triangle ABC$ (right-hand picture) and explain how you did it.
(c) If $AC = \sqrt{3}$ inches long, how long are $\overline{AB}$ and $\overline{BE}$?

8. Three squares of Origami paper are piled one on top of the other. The corners of the smallest square touch the midpoints of the sides of the black square. The black square fits on the largest square the same way.



(a) If the largest square measures 10 cm on the side, what are the lengths of the sides of the other two squares?
(b) If the smallest square measures 5 cm along its side, what is the length of its diagonal? What is the length of the largest square's diagonal?

Hints

Hint to problem 2. The three angles are the same. Their sum is $180°$.

Hint to problem 8. The diagonals of the black square match the sides of the largest square.

Answers

Measuring their angles and sides

1. See solutions.

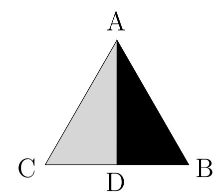
2. See solutions.

3. See solutions.

4. $HB = \frac{1}{2}$ inch.

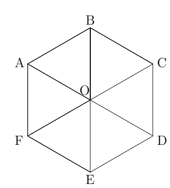
5. $AH = \sqrt{\frac{3}{4}}$ (That can also be written as $AH = \frac{\sqrt{3}}{2}$.)
$DF = \sqrt{2}$

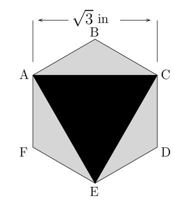
Getting to know them well



6. (a) If $DB = 1$, then $AB = 2$ and $AD = \sqrt{3}$.
(b) If $AB = x$, $DB = \frac{x}{2}$ and $AD = \frac{\sqrt{3}}{2}$.
(c) If $DC = x$, $AC = 2x$ and $AD = x\sqrt{3}$.
(d) If $AD = 5$, then $DB = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$ and $AB = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$.
(e) For *any* size of $\overline{AB}$, $\frac{AD}{DB} = \sqrt{3}$ and $\frac{AB}{DB} = 2$.

These ratios come up often and are worth remembering.

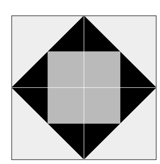




7. (a) The lefthand picture shows that each interior angle of the regular hexagon is composed of two 60° angles, so $m\angle ABC = m\angle BCD = m\angle CDE = \ldots = 120°$.
(b) $m\angle ABC = 120°$, leaving 60° for the sum of angles BAC and BCA. Symmetry shows that those two angles are congruent, so $m\angle BAC = m\angle BCA = 30°$.

Another way of computing the size of $\angle BAC$ is to notice that $\angle BAC + \angle EAC + \angle EAC = 30°$.
(c) $\overline{AB}$ is 1″ long, and $\overline{BE}$ is 2″ long.

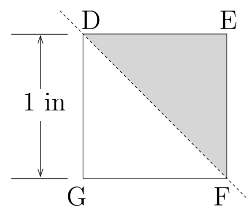
Remember! $\frac{\text{the diagonal of a square}}{\text{the side of the same square}} = \sqrt{2}$.

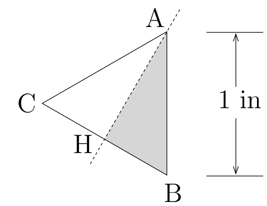


8. (a) If the largest square's side measures 10 cm, then the black square's side is $\frac{10}{\sqrt{2}}$, or $5\sqrt{2}$ and the smallest square's side is $\frac{5\sqrt{2}}{\sqrt{2}}$, or 5, half that of the largest square.
(b) If the smallest square's side is 5 cm, its diagonal is $5\sqrt{2}$ and the largest square's diagonal is $10\sqrt{2}$.

Solutions

Measuring their angles and sides





1. A square's corners are right angles, so $m\angle DEF = 90°$. Because the dotted line is a line of symmetry, $\angle FDE$ and $\angle FDG$ match, so each must be half of the right angle. That makes $m\angle FDE = 45°$. The same is true of $\angle EFD$ and $\angle GFD$, which means that $\angle EFD$ also measures 45°.

2. The sum of the angles in any triangle is 180°. The three angles in $\triangle ABC$ are all the same, so each must be 60°.

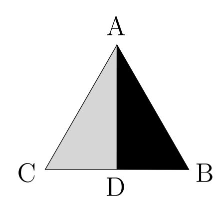
3. You already know that $m\angle HBA = 60°$. Angles BAH and CAH are each half of $\angle BAC$, so $m\angle BAH = 30°$. There are two ways to see that angle AHB measures 90°. The sum of interior angles in a triangle is 180°, and the sum of $\angle BAH$ and $\angle HBA$ is 90°, leaving 90° for $\angle AHB$. Or you could argue that $m\angle CHB = 180°$ and the line of symmetry splits it equally, making $m\angle AHB = 90°$.

4. All sides of the equilateral triangle are equal. Because the $\overline{HB}$ is half of side $\overline{CB}$, $HB = \frac{1}{2}$ inch.

5. $AH^2 = AB^2 - BH^2 = 1 - (\frac{1}{2})^2 = \frac{3}{4}$ so
$AH = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$.

$DF^2 = DE^2 + EF^2 = 2$ so
$DF = \sqrt{2}$.

Getting to know them well



6. $AB^2 = AD^2 + DB^2$ and $AC^2 = AD^2 + DC^2$.
Because $\angle ADB$ is a right angle, we also know that $\triangle ABD$ is a 30°-60°-90° triangle, so $DB = \frac{1}{2}AB = \frac{1}{2}CB = DC$.

(a) If $DB = 1$, then $CB = 2$ so $AB = 2$.
By the Pythagorean theorem, $AD = \sqrt{3}$.

(b) If $AB = x$, $DB = \frac{x}{2}$ and $AD = \frac{\sqrt{3}}{2}$.

(c) If $DC = x$, $AC = 2x$ and $AD = x\sqrt{3}$.

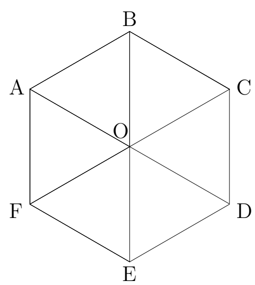
(d) Part (c) shows that if $DB = x$, then $AD = x\sqrt{3}$, so if $AD = 5$, then $DB = \frac{5}{\sqrt{3}}$. $AB = 2DB$, so $AB = \frac{10}{\sqrt{3}}$.

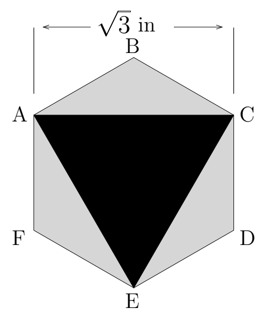
If you care about "rationalizing the denominator," then $DB = \frac{5\sqrt{3}}{3}$ and $AB = \frac{10\sqrt{3}}{3}$.

(e) Part (c) shows that if $DB = x$, then $AD = x\sqrt{3}$, so for *any* size of $\overline{AB}$, $\frac{AD}{DB} = \frac{x\sqrt{3}}{x} = \sqrt{3}$. Also, for *any* size of $\overline{AB}$, $\frac{AB}{DB} = 2$.

Because these results apply for *all* 30°-60°-90° triangles, and because these triangles are important, these ratios are worth remembering.

7.





(a) The picture on the left shows that each interior angle of the regular hexagon is composed of two 60° angles, so $m\angle ABC = m\angle BCD = m\angle CDE = \ldots = 120°$.

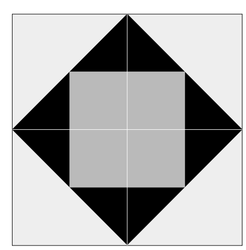
(b) $m\angle ABC = 120°$, leaving 60° for the sum of angles BAC and BCA. Symmetry shows that those two angles are congruent, so $m\angle BAC = m\angle BCA = 30°$.

Another way of computing the size of $\angle BAC$ is to notice that $\angle BAC + \angle EAC + \angle EAC = 30°$.

(c) Looking at *both* pictures shows that $\overline{AC}$ bisects the two equilateral triangles, $\triangle ABO$ and $\triangle BCO$, and is twice the altitude of those triangles. That altitude must therefore be $\frac{AC}{2} = \frac{\sqrt{3}}{2}$ inches. That makes $\overline{AB}$ 1″ long, and $\overline{BE}$ 2″ long.

8. The key feature of the picture is that the diagonal of each square is the same length as the side of the next larger square on which it lies.

Remember! $\frac{\text{the diagonal of a square}}{\text{the side of the same square}} = \sqrt{2}$.



(a) If the largest square's side measures 10 cm, so does the black square's diagonal. Because $\frac{\text{diagonal}}{\text{side}} = \sqrt{2}$, the black square's side (and smallest square's diagonal!) is $\frac{10}{\sqrt{2}}$, or $5\sqrt{2}$. The same reasoning shows that the smallest square's side is $\frac{5\sqrt{2}}{\sqrt{2}}$, or 5, half that of the largest square.

(b) This problem just reverses the reasoning. If the smallest square measures 5 cm along its side, its diagonal is $5\sqrt{2}$. The side of the largest square is 10, so its diagonal is $10\sqrt{2}$.

