What are arithmetic sequences and series *for*? Problem

When a ball is thrown straight up in the air, it starts out fast, gradually slows until it reaches its maximum height, and then it starts coming down again.

A *change in speed* can be described as a *change in distance traveled during a unit of time*.

Each unit of time, the ball goes up less than it did during the previous unit. Remarkably, the amount of decrease from one unit of time to the next is constant!

The amount of change depends on gravity, which is different on different planets.

For example, suppose that the ball moves straight up $52''$ in the first tenth of a second after it is thrown. In the next $\frac{1}{10}$ sec, it won't rise as much. Suppose it rises only $48''$—that is, $4''$ less than it did in the first $\frac{1}{10}$ sec. Then, each tenth of a second, it will again rise $4''$ less than it did the tenth of a second before. Here are two questions one might ask about such a situation.

For our planet, the $4''$ decrease of upward movement every tenth of a second is very close to the true pull of gravity.

- How long will it be before the ball starts coming back down?
- How high will it go?

One way to figure out answers to questions like these is to list how far the ball travels each tenth of a second.

Time after release from hand	1st 10th of a sec	2nd 10th of a sec	3rd 10th of a sec	...
Distance traveled during that tenth of a second	52 in.	48 in.	44 in.	...

1. Create a table like this and fill it in for each tenth of a second for the first 16 tenths of a second.

In this problem, you are creating an **arithmetic sequence**, a sequence of numbers that differ by a constant amount (in this case -0.8) from term to term.

2. What total distance does the ball travel upward during the first 0.2 sec?

Here you are *summing* an **arithmetic series**.

3. How far does the ball travel during the first second?

4. At what time does the ball no longer go up?

Here, you are finding which term in an arithmetic sequence has a particular value.

5. From the table, how high does the ball go at its highest?

Again, to solve this you are **summing a series**.

6. What do the negative numbers in your table mean?

7. From the time that the ball has reached its highest, how long will it be before the ball reaches your hand again?

Try to figure this out without extending your table.

Hints

Hint to problem 6. Think about where the ball is and what it is doing just a moment earlier. Think what must happen next. Then explain the number (both its *size* and its *sign*) in terms of what the ball is doing.

Hint to problem 7. Compare the distances traveled in the two tenths of a second just before and after the very top of the ball's trip.

Answers

1. See table below.

Time	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$	$\frac{11}{10}$	$\frac{12}{10}$	$\frac{13}{10}$	$\frac{14}{10}$	$\frac{15}{10}$	$\frac{16}{10}$
Dist	$52''$	$48''$	$44''$	$40''$	$36''$	$32''$	$28''$	$24''$	$20''$	$16''$	$12''$	$8''$	$4''$	$0''$	$-4''$	$-8''$

2. During the first 0.2 sec, the ball travels upward 100 inches.

3. During the first 1 sec, the ball travels upward 340 inches.

4. Some time during the fourteenth $\frac{1}{10}$ of a second the ball doesn't go up any more.

5. By then, using the numbers above, it has gone up $364''$.

6. The negative numbers in the table mean that the ball is falling back down. The amount of falling increases each tenth of a second.

7. It will take 1.4 sec.

Solutions

1.

Time	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$	$\frac{11}{10}$	$\frac{12}{10}$	$\frac{13}{10}$	$\frac{14}{10}$	$\frac{15}{10}$	$\frac{16}{10}$
Dist	$52''$	$48''$	$44''$	$40''$	$36''$	$32''$	$28''$	$24''$	$20''$	$16''$	$12''$	$8''$	$4''$	$0''$	$-4''$	$-8''$

2. By the end of 0.2 sec, the ball has gone $52'' + 48''$ for a total of $100''$.

3. By the end of 1 sec, the ball has traveled upward $52'' + 48'' + 44'' + 40'' + 36'' + 32'' + 28'' + 24'' + 20'' + 16''$. This is quite a nuisance to add up unless you notice that

$$\begin{array}{rrrrrrrrrrrrrrrrrrrr} & 52 & + & 48 & + & 44 & + & 40 & + & 36 & + & 32 & + & 28 & + & 24 & + & 20 & + & 16 \\ + & 16 & + & 20 & + & 24 & + & 28 & + & 32 & + & 36 & + & 40 & + & 44 & + & 48 & + & 52 \\ \hline & 68 & + & 68 & + & 68 & + & 68 & + & 68 & + & 68 & + & 68 & + & 68 & + & 68 & + & 68 \end{array}$$

In other words, the sum of the first term and last term, $(52 + 16) = 68$, times the number of terms (10), is *twice* the sum that you want. So, an easy way to compute $52 + 48 + 44 + 40 + 36 + 32 + 28 + 24 + 20 + 16$ is to compute $(52 + 16) \times 10 \div 2$. This is 340.

4. During the fourteenth $\frac{1}{10}$ of a second, the ball rises $0''$. This doesn't mean that the ball is sitting still for a tenth of a second! It does mean that the ball's height at the beginning and end of that interval of time is the same (the change in height from beginning to end is 0). In between, it goes up a bit more and comes back down.

5. Using the numbers in the table, the ball will have gone a total of $(52 + 0) \times 14 \div 2$, which is 364 inches up. In fact, making the reasonable (and correct) assumption that the fourteenth $\frac{1}{10}$-second is split evenly between "up" and "down," the total distance the ball travels up is $2''$ more than one can read off the table—366 inches at its highest. The way this question is worded, either 364 or 366 is a reasonable answer.

6. The negative numbers in the table mean that the ball is falling back down. The amount of falling increases by $4''$ each tenth of a second.

7. The distance traveled each $\frac{1}{10}$-second on the way down exactly mirrors the distance traveled on the way up, so the time it takes to get downis the same as the time it took to get up. It took the ball 1.4 sec to reach its highest point, so it will take another 1.4 sec to get back down.

These figures come from the table. If we account for the "extra" travel up during the fourteenth $\frac{1}{10}$-second, it takes 1.45 sec to go up and the same to get down.

