Inventing a formula for arithmetic sequences Problem

Here are the first five terms of an **arithmetic sequence**: $4, 7, 10, 13, 16, \ldots.$

Definition: In an **arithmetic sequence**, the difference between any two successive numbers is the same.

1. What are the next three terms?

2. How much greater is the second term than the first?

3. (a) How much greater is the third term than the first?
(b) How much greater is T_4 (the fourth term) than T_1?
(c) How much greater is T_7 (the seventh term) than T_4?

4. What must the value of T_{100} be?

You can figure this out without knowing the terms in between, and without any special formula.

Here are the first three terms of a different arithmetic sequence: $53, 49.2, 45.4, \ldots.$

5. What is the **rate of change**, $T_2 - T_1$, in this sequence?

Definition: The **rate of change** is one name for the constant difference between any two successive terms of an arithmetic sequence. Pay attention to the *sign* of the rate of change.

6. What are the next three terms?

7. (a) Compute $T_6 - T_1$. Explain why it is 5 times $T_2 - T_1$.
(b) Explain how it is possible to compute $T_8 - T_1$ without computing T_8. Compute $T_8 - T_1$.
(c) Compute $T_{26} - T_{21}$ without knowing either T_{26} or T_{21}.

8. What is the value of T_{100}?

A new arithmetic sequence begins this way: $a,\ a+b,\ a+2b,\ \ldots$

9. What is the rate of change in this sequence?

10. What are terms T_1, T_2, T_3, T_4, and T_5?

11. (a) Compute T_{100}.
(b) Compute T_{6000}.
(c) Compute T_n.

12. (a) Compute $T_9 - T_1$.
(b) Compute $T_{100} - T_1$.
(c) Compute $T_{100} - T_{83}$.
(d) Compute $T_n - T_k$.

13. Explain in words (or as a formula) how to find T_n of *any* arithmetic sequence if you know T_1 and b (rate of change).

Hints

Hint to problem 1. Try problem 2. Then return to this one.

Hint to problem 4. How many "steps" away from T_1 is T_{100}?

Hint to problem 7(a). How many steps away from T_1 is T_6?

Hint to problem 9. How have you computed the rate of change in earlier problems?

Hint to problem 12(a). How many steps from T_1 is T_9? What is the size of each step?

Hint to problem 13. Look carefully at your answers to problem 12.

Answers

1. The next three terms are 19, 22, and 25.

2. The second term is 3 greater than the first.

3. (a) The third term is 6 greater than the first.
(b) T_4 is 9 greater than T_1.
(c) T_7 is 9 greater than T_4.

4. $T_{100} = 301$.

5. The rate of change is -3.8.

6. The next three terms are 41.6, 37.8, and 34.

7. (a) $T_6 - T_1 = -19$. The change in *each* step is -3.8, so in five steps, the total change is -19.
(b) It takes 7 steps to get from T_1 to T_8, and each step adds -3.8. So $T_8 - T_1 = 7 \times (-3.8) = -26.6$.
(c) $T_{26} - T_{21} = -19$.

8. $T_{100} = -323.2$.

9. The rate of change is b.

10. $T_1 = a$
$T_2 = a + b$
$T_3 = a + 2b$
$T_4 = a + 3b$
$T_5 = a + 4b$

11. (a) $T_{100} = a + 99b$
(b) $T_{6000} = a + 5999b$
(c) $T_n = a + (n-1)b$

12. (a) $T_9 - T_1 = 8b$
(b) $T_{100} - T_1 = 99b$
(c) $T_{100} - T_{83} = 17b$
(d) $T_n - T_k = (n-k)b$

13. $T_n - T_1 = (n-1)b$, so $T_n = T_1 + (n-1)b$.
For example, to find the 20^{th} term in the sequence, add $19b$ to the first term in the sequence.

Solutions

1. In the arithmetic sequence $4, 7, 10, 13, 16, ...$, each term is 3 greater than the preceding term, so the next three terms must be 19, 22, and 25.

2. The second term is 3 greater than the first.

3. (a) The third term is 6 greater than the first.
(b) T_4 is 9 greater than T_1.
(c) T_7 is 9 greater than T_4.

4. Each term is 3 more than the term before it, so ninety-nine 3s must be added to T_1 to produce T_{100}.
Therefore, $T_{100} = T_1 + 99 \cdot 3 = 4 + 297 = 301$.

5. In the arithmetic sequence, $53, 49.2, 45.4, ...$, the rate of change is $49.2 - 53$, or $45.4 - 49.2$, both of which are -3.8.

6. To find the fourth term, add -3.8 to the third term.
Likewise, $T_5 = T_4 + -3.8$, and $T_6 = T_5 + -3.8$.
So, terms T_4, T_5, and T_6 are 41.6, 37.8, and 34.

$45.4 + -3.8 = 41.6$

7. (a) $T_6 = 34$ and $T_1 = 53$ so $T_6 - T_1 = 34 - 53 = -19$. In each step of this arithmetic sequence, another -3.8 is added, so in five steps, $5 \times (-3.8) = -19$ is added.
(b) It takes 7 steps to get from T_1 to T_8, and each step adds -3.8. So $T_8 - T_1 = 7 \times (-3.8) = -26.6$.
(c) T_{26} and T_{21} are separated by the same number of steps as T_6 and T_1, so $T_{26} - T_{21}$ is the same as $T_6 - T_1$. Therefore, $T_{26} - T_{21} = -19$.

8. $T_{100} = T_1 + 99 \times (-3.8) = 53 - 376.2 = -323.2$.

9. If a and $a + b$ are the first two terms of an arithmetic sequence, then b is the rate of change.

$(a + b) - a = b$

10.

$T_1 = a$	(given)
$T_2 = a + b$	(given)
$T_3 = a + 2b$	(given)
$T_4 = a + 3b$	$T_4 = T_3 + b = (a + 2b) + b = a + 3b$
$T_5 = a + 4b$	$T_5 = T_4 + b = T_3 + 2b = T_2 + 3b = T_1 + 4b$

Because the rate of change is b, add b to each term to get the next term.

11. (a) $T_{100} = a + 99b$
(b) $T_{6000} = a + 5999b$
(c) $T_n = a + (n - 1)b$
Here's the general pattern:

$$T_n = T_{(n-1)} + b = T_{(n-2)} + 2b = T_{(n-3)} + 3b = T_{(n-4)} + 4b = \ldots = T_1 + (n-1)b$$

12. This problem applies the general pattern found in the two previous problems.

(a) $T_9 - T_1 = 8b$.

(b) $T_{100} - T_1 = 99b$.

(c) $T_{100} - T_{83} = 17b$.

(d) $T_n - T_k = (n - k)b$.

13. $T_n - T_1 = (n-1)b,$ so $T_n = T_1 + (n-1)b.$

In words, this says: "To find the n^{th} term in an arithmetic sequence, add $(n-1)b$ to the first term of that sequence, where b is the constant rate of change."

For example, to find the 20^{th} term in the sequence, add $19b$ to the first term in the sequence.

