Which fractions become terminating decimals?

When you look at a common fraction—a fraction like $\frac{1}{2}$ or $\frac{5}{6}$ —can you tell right away whether it will have a terminating decimal or a repeating decimal?

In this problem sequence, you will develop a general rule that will let you recognize easily which repeat and which terminate.

1. You probably already know several examples of each type.
  (a) List five common fractions that you believe have *terminating* decimal expansions.
  (b) List three common fractions that you believe have *repeating* decimal expansions.

You may perform an experiment, if you like, to find new examples for this problem.

2. You might already have an idea about how to tell which fractions' decimals will terminate and which will repeat. If so, jot it down now.

If you don't have a conjecture now, don't worry. Either way, you'll get a chance to extend your ideas soon.

3. To develop or test conjectures, collect some more data. Find the decimal expansion of each of these fractions.

For most of this task, a calculator will help. For some of these, the calculator may not give you as much information as you want, and you may need to calculate the old-fashioned way!

Fraction	**Expansion**
$\frac{1}{2}$	
$\frac{1}{3}$	
$\frac{1}{4}$	
$\frac{1}{5}$	
$\frac{1}{6}$	
$\frac{1}{7}$	$0.\overline{142857}$
$\frac{1}{8}$	0.125
$\frac{1}{9}$	
$\frac{1}{10}$	
$\frac{1}{11}$	
$\frac{1}{12}$	
$\frac{1}{13}$	
$\frac{1}{14}$	
$\frac{1}{15}$	
$\frac{1}{16}$	
$\frac{1}{17}$	
$\frac{1}{18}$	
$\frac{1}{19}$	
$\frac{1}{20}$	

The bar above the digits $\overline{142857}$ means that these digits repeat forever: 0.142857142857142857142857142857....

4. Which of these fractions have terminating decimals? What *else* do they have in common?

5. Based on your rule, stated in problem 4, guess three more terminating fractions with denominators between 20 and 100. Turn them into decimals to check that they terminate.

6. Turn the following decimals into fractions:

(a) 0.1
(b) 0.33
(c) 0.541

7. Compare the denominators of the three fractions in problem 6 with the denominators of the terminating decimals you've found before. What is common among all of them?

8. First, turn these decimals into fractions with denominators of 10, 100, or 1000. Then divide the numerator and denominator by *prime* factors until the fractions are completely reduced. *Keep track of the factors you divide by.*

Of course, to reduce a fraction properly, you must divide the numerator and denominator by the *same* number.

(a) 0.075
(b) 0.4
(c) 0.06
(d) 0.175
(e) 0.45
(f) 0.176

9. (a) What factors did you use in reducing the fractions?

Why *only* those factors?

(b) What are the factors of the *reduced* denominators?

Why *only* those factors?

10. Now you should be able to answer the question that titles this problem set. Finish the following two sentences:

(a) The decimal expansion of a common fraction terminates if that common fraction...

(b) With that kind of common fraction, the decimal expansion terminates because...

Hints

Hint to problem 4. Look at their denominators.

Hint to problem 9. What are the prime factors of 10, 100, 1000, and so on?

Solutions

1. (a) Familiar common fractions that have terminating decimal expansions include:

$\frac{1}{2} = 0.5 \quad \frac{1}{10} = 0.1 \quad \frac{1}{5} = 0.2$

$\frac{1}{4} = 0.25 \quad \frac{1}{100} = 0.01 \quad \frac{3}{10} = 0.3$

$\frac{3}{4} = 0.75 \quad \frac{1}{1000} = 0.001 \quad \frac{2}{5} = 0.4$

(b) Familiar common fractions that have non-terminating (repeating) decimal expansions include:

$\frac{1}{3} = 0.\overline{3} \quad \frac{1}{9} = 0.\overline{1} \quad \frac{1}{6} = 0.1\overline{6}$

$\frac{2}{3} = 0.\overline{6} \quad \frac{2}{9} = 0.\overline{2} \quad \frac{5}{6} = 0.8\overline{3}$

2. Answers may vary.

3.

Fraction	**Expansion**
$\frac{1}{2}$	0.5
$\frac{1}{3}$	$0.\overline{3}$
$\frac{1}{4}$	0.25
$\frac{1}{5}$	0.2
$\frac{1}{6}$	$0.1\overline{6}$
$\frac{1}{7}$	$0.\overline{142857}$
$\frac{1}{8}$	0.125
$\frac{1}{9}$	$0.\overline{1}$
$\frac{1}{10}$	0.1
$\frac{1}{11}$	$0.\overline{09}$
$\frac{1}{12}$	$0.08\overline{3}$
$\frac{1}{13}$	$0.\overline{076923}$
$\frac{1}{14}$	$0.0\overline{714285}$
$\frac{1}{15}$	$0.0\overline{6}$
$\frac{1}{16}$	0.0625
$\frac{1}{17}$	$0.\overline{0588235294117647}$
$\frac{1}{18}$	$0.0\overline{5}$
$\frac{1}{19}$	$0.\overline{052631578947368421}$
$\frac{1}{20}$	0.05

4. The fractions (in this table) that have terminating decimals are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{16}$, and $\frac{1}{20}$. The other thing that they have in common is that the only factors their denominators have are 2 and 5.

5. Here are some of the many possible correct answers.
$\frac{1}{25} = 0.04$; $\frac{1}{40} = 0.025$; $\frac{1}{80} = 0.0125$; $\frac{1}{125} = 0.008$;
$\frac{3}{8} = 0.375$; $\frac{17}{25} = 0.68$; $\frac{39}{40} = 0.975$.

6. $0.1 = \frac{1}{10}$; $0.33 = \frac{33}{100}$; $0.541 = \frac{541}{1000}$.

7. The denominators of these fractions are 10, 100, and 1000, which also have factors 2 and 5.

8. (a) $0.075 = \frac{75}{1000} = \frac{3}{40}$

(b) $0.4 = \frac{4}{10} = \frac{2}{5}$

(c) $0.06 = \frac{6}{100} = \frac{3}{50}$

(d) $0.175 = \frac{175}{1000} = \frac{7}{40}$

(e) $0.45 = \frac{45}{100} = \frac{9}{20}$

(f) $0.176 = \frac{176}{1000} = \frac{22}{125}$

9. (a) The denominators of the initial fractions were always powers of 10, whose only (prime) factors are 2 and 5. Therefore 2 and 5 are the only prime factors that can reduce these fractions.

(b) The reduced denominators still have 2 and 5 as their only prime factors, because those were the only factors the original denominators had. The denominators of reduced fractions *can not possibly* have any factors other than 2 and 5 (such as 3, 7, 11, and so on) because these numbers are not factors of 10, 100, 1000, or any other powers of 10.

10. (a) The decimal expansion of a common fraction terminates if that common fraction has a denominator whose only prime factors are 2 and 5.

(b) With that kind of common fraction, the decimal expansion terminates because you can multiply both numerator and denominator by a whole number (whose only prime factors are 5 and 2!) to get fractions with denominators 10, 100, 1000, and so on. All such fractions translate straight into decimals that terminate.

