The binomial theorem 1
Problem

1 Finding the Binomial Theorem

Have you heard of the Binomial Theorem? It's got a fancy name, so it must be important, right? The Binomial Theorem provides a formula for the expansion of powers of a binomial expression like $a+b$. To get a feel for the *need* of the theorem, what the theorem might say, and why anyone should care about it, you can look at a few relatively small powers of $a+b$.

1. Expand each of the following binomial powers. Here's something to think about as you work: How can you get one answer from a previous one?
  (a) $(a+b)^2$
  (b) $(a+b)^3$
  (c) $(a+b)^4$
  (d) $(a+b)^5$
  (e) What patterns do you notice in the coefficients of these polynomials?

Aren't you glad you weren't asked to expand and simplify $(a+b)^6$? Well, you will in a moment, but first, ...

2. *Predict* what the expansion of $(a+b)^6$ will look like.

3. Use the expansion of $(a+b)^5$ to expand $(a+b)^6$ and check to see if your prediction was correct.

Enough suspense! Are you ready for a quick method that determines the expansion of $(a+b)^n$ for *any* positive integer n? You may have already guessed the punch line:

THE BINOMIAL THEOREM:

$$(a+b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \cdots + \binom{n}{n-1}a^{n-1}b^1 + \binom{n}{n}a^nb^0.$$

The notation $\binom{n}{k}$ represents the quantity $\frac{n!}{k!(n-k)!}$. For example, $\binom{5}{3} = \frac{5!}{3!2!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(2\cdot1)} = 10$.

The binomial theorem allows you to expand powers of a binomial expression quickly, especially if you know the values of the binomial coefficients, $\binom{n}{k}$. For example,

On many graphing calculators, these coefficients are listed as `nCr`. In order to compute $\binom{8}{3}$, type `8 nCr 3`.

$$(a+b)^5 = \binom{5}{0}a^0b^5 + \binom{5}{1}a^1b^4 + \binom{5}{2}a^2b^3 + \binom{5}{3}a^3b^2 + \binom{5}{4}a^4b^1 + \binom{5}{5}a^5b^0$$

$$= \frac{5!}{0!5!}b^5 + \frac{5!}{1!4!}a^1b^4 + \frac{5!}{2!5!}a^2b^3 + \frac{5!}{3!5!}a^3b^2 + \frac{5!}{4!5!}a^4b^1 + \frac{5!}{5!0!}a^5$$
$$= b^5 + 5ab^4 + 10a^2b^3 + 10a^3b^2 + 5a^4b + a^5$$
$$(\text{or } \; a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5)$$

Do you recognize the coefficients in the expansion of $(a+b)^5$? They're the entries in row 5 of *Pascal's triangle*.

2 Applying the Binomial Theorem

Here are a few examples to illustrate how much time can be saved using the theorem instead of multiplying everything out by hand.

1. (a) Expand $(a+b)^6$ using the binomial theorem, being sure you get the same answer as in problem 5.
(b) Expand $(a+b)^7$ using the binomial theorem.
(c) Expand $(x+2y)^5$ using the binomial theorem.
(d) Expand $(3x-2y)^7$ using the binomial theorem.

2. (a) What is the coefficient of $a^{169}b^6$ in the expansion of $(a+b)^{175}$?
(b) What value of k, other than 169, will make the equation $\binom{175}{169} = \binom{175}{k}$ true?
(c) What other term in the expansion of $(a+b)^{175}$ has the same coefficient as $a^{169}b^6$?

Think about these problems *without* actually determining the value of 175!. After all, it would take a *very* long time by hand and many computer algebra systems wouldn't be able to do it.

Hints

1 Finding the Binomial Theorem

Hint to problem 1. While you aren't *prohibited* from using technology (in the form of a computer algebra system, for example), it's really not necessary. Just *carefully* multiply everything out and simplify. On subsequent calculations, be sure to make use of previous results.

Hint to problem 2. Look back at the results of problem 1. Is there a pattern to the coefficients? Where have you seen the coefficients before?

Hint to problem 3. Multiply your expansion for $(a+b)^5$ by $(a+b)$, being sure to carefully use the distributive property.

2 Applying the Binomial Theorem

Hint to problems 1(a) and (b). Determine the necessary binomial coefficients using the formula.

Hint to problem 1(c). Use the expansion of $(a+b)^5$ with appropriate substitutions for a and b.

Hint to problem 1(d). Use the expansion of $(a+b)^7$ with appropriate substitutions for a and b.

Hint to problem 2(a). Think how you might compute $\frac{175!}{169!6!}$ without attempting to determine 175!. Are there some terms you can cancel from both the numerator and denominator?

Hint to problem 2(b). What does $\binom{175}{k}$ "look like?"

Hint to problem 2(c). *Don't* try to expand $(a+b)^{175}$! Think about what the binomial theorem says about these coefficients.

Answers

1 Finding the Binomial Theorem

1. (a) $(a+b)^2 = a^2 + 2ab + b^2$
(b) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(c) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
(d) $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

2. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

3. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

2 Finding the Binomial Theorem

1. (a) $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
(b) $a^7 + 14a^6b + 84a^5b^2 + 280a^4b^3 + 560a^3b^4 + 672a^2b^5 + 448ab^6 + 128b^7$
(c) $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$
(d) $2187x^7 - 10206x^6y + 20412x^5y^2 - 22680x^4y^3 + 15120x^3y^4 - 6048x^2y^5 + 1344xy^6 - 128y^7$

2. (a) 36582584325
(b) $\binom{175}{6}$
(c) a^6b^{169}

Solutions

1 Finding the Binomial Theorem

1. (a) $(a+b)^2 = a^2 + 2ab + b^2$

(b) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(c) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

(d) $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

(e) The coefficients in the expansion of $(a+b)^n$ seem to be the entries in row n of Pascal's Triangle.
Alternatively, the coefficient of each term (other than a^n and b^n) in the expansion of $(a+b)^n$ is the sum of two coefficients in the expansion of $(a+b)^{n-1}$. For example, the coefficient of $a^k b^{5-k}$ (when $0 < k < 5$) in the expansion of $(a+b)^5$ is the sum of the coefficients of $a^k b^{4-k}$ and $a^{k-1}b^{5-k}$ in the expansion of $(a+b)^4$.

Specifically, the coefficient of a^2b^3 in the expansion of $(a+b)^5$ is the sum of the coefficients of a^2b^2 and a^1b^3 in the expansion of $(a+b)^4$. This is due to the fact that the term a^2b^3 in the expansion of $(a+b)^5$ is obtained by multiplying $4ab^3$ by a and by multiplying $6a^2b^2$ by b and adding the results.

2. The coefficients correspond to those numbers in the 6th row of Pascal's triangle.
Alternatively, the coefficient of $a^k b^{6-k}$ (when $0 < k < 6$) in the expansion of $(a+b)^6$ is the sum of the coefficients of $a^k b^{5-k}$ and $a^{k-1}b^{6-k}$ in the expansion of $(a+b)^5$.
However you look at it, this results in the prediction that $(a+b^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

Teachers note: This prediction is not obvious, especially if students are not familiar with Pascal's Triangle. You might want to provide some additional guidance.

3. We know from problem 1 that $(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3 + 5ab^4 + b^5$. Therefore, we know that

$$\begin{aligned}
(a+b)^6 &= (a+b)^5(a+b) \\
&= (a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5)(a+b) \\
&= a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5 \\
&\quad + a^5y + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6 \\
&= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.
\end{aligned}$$

2 Applying the Binomial Theorem

1. (a) $(a+b)^6 = \binom{6}{0}a^0b^6 + \binom{6}{1}a^1b^5 + \binom{6}{2}a^2b^4 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^4by^2 + \binom{6}{5}a^5b^1 + \binom{6}{6}a^6b^0$, which simplifies to

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

(b) $(a+b)^7 = \binom{7}{0}a^0b^7 + \binom{7}{1}a^1b^6 + \binom{7}{2}a^2b^5 + \binom{7}{3}a^3b^4 + \binom{7}{4}a^4by^3 + \binom{7}{5}a^5b^2 + \binom{7}{6}a^6b^1 + \binom{7}{7}a^0b^7$, which equals

$$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

(c) Letting $a = x$ and $b = 2y$ in the expansion of $(a+b)^5$, we have

$$\begin{aligned}(x+2y)^5 &= \binom{5}{0}x^0(2y)^5 + \binom{5}{1}x^1(2y)^4 + \binom{5}{2}x^2(2y)^3 \\ &\quad + \binom{5}{3}x^3(2y)^2 + \binom{5}{4}x^4(2y)^1 + \binom{5}{5}x^5(2y)^0 \\ &= \frac{5!}{0!5!}(2y)^5 + \frac{5!}{1!4!}x^1(2y)^4 + \frac{5!}{2!5!}x^2(2y)^3 \\ &\quad + \frac{5!}{3!5!}x^3(2y)^2 + \frac{5!}{4!5!}x^4(2y)^1 + \frac{5!}{5!0!}x^5 \\ &= 32y^5 + 80ab^4 + 80a^2b^3 + 40a^3b^2 + 10a^4b + a^5 \\ &\quad (\text{or} \quad a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5)\end{aligned}$$

(d) Let $a = 3x$ and $b = -2y$. Then

$$\begin{aligned}(3x-2y)^7 &= \binom{7}{0}(3x)^0(2y)^7 + \binom{7}{1}(3x)^1(2y)^6 + \binom{7}{2}(3x)^2(2y)^5 + \binom{7}{3}(3x)^3(2y)^4 \\ &\quad + \binom{7}{4}(3x)^4(2y)^3 + \binom{7}{5}(3x)^5(2y)^2 + \binom{7}{6}(3x)^6(2y)^1 + \binom{7}{7}x^7(2y)^0 \\ &= (3x)^0(-2y)^7 + 7(3x)^1(-2y)^6 + 21(3x)^2(-2y)^5 + 35(3x)^3(-2y)^4 \\ &\quad + 35(3x)^4(-2y)^3 + 21(3x)^5(-2y)^2 + 7(3x)^6(2y) + (3x)^7(-2y)^0 \\ &= 2187x^7 - 10206x^6y + 20412x^5y^2 - 22680x^4y^3 \\ &\quad + 15120x^3y^4 - 6048x^2y^5 + 1344xy^6 - 128y^7.\end{aligned}$$

2. (a) The coefficient of $a^k b^{175-k}$ in the expansion of $(a+b)^{175}$ is $\binom{175}{k}$, so the coefficient of $a^{169}b^6$ is

$$\begin{aligned}\binom{175}{169} &= \frac{175!}{169!6!} \\ &= \frac{(175 \cdot 174 \cdot 173 \cdot 172 \cdot 171 \cdot 170) \cdot 169!}{169!6!} \\ &= \frac{175 \cdot 174 \cdot 173 \cdot 172 \cdot 171 \cdot 170}{6!} \\ &= \frac{26339460714000}{720} \\ &= 36582584325.\end{aligned}$$

(b) $\binom{175}{169} = \frac{175!}{169!6!} = \frac{175!}{6!169!} = \binom{175}{6}$,

(c) Since $\binom{175}{6}$ is the coefficient of $a^6 b^{169}$ in the expansion of $(a+b)^{175}$, we know that $a^{169}b^6$ and $a^6 b^{69}$ have the same coefficients.

