Where's the (decimal) point?—part 1

When required to compute (by hand) products like 2.38×0.001045 and 23800000×1045000, many people ignore the decimals or zeros, compute the product 1045×238, which equals $248,710$, and then put the decimals or zeros back carefully in order to get the answers 0.0024871 and $24,871,000,000,000$ (or 2.4871×10^{-3} and 2.4871×10^{13}, if you prefer scientific notation).

1. Carefully explain how to get from 248,710—the result of the product 1045×238—to the product 2.38×0.001045. That is, explain *how* to place the decimal point correctly.

2. Carefully apply the method you just used to compute the product 2.38×0.001045 to explain a method for multiplying *any* two decimal numbers. Be sure the method works for any decimal number.

3. How do you determine how many zeros you need to put back in order to get from 327,310 (the product of 142 and 2305) to the answer to 14200000×2305000?

4. Now explain *why* the "ignore the decimals (or zeros), multiply the integers, then put the decimal (or zeros) back" methods you just described work.

Numbers like the ones you multiplied in the previous problems are often written in scientific notation. In order to use the methods you've been working with on numbers written in scientific notation, you first need to rewrite the numbers.

Numbers like $23,800,000$ and 0.001045 are often written in scientific notation, instead (as 2.38×10^7 and 1.045×10^{-3}).

5. Express each of the following numbers as an integer times a power of 10.
(a) 1.6×10^{11}
(b) 2.3×10^5
(c) 1.05×10^{-3}
(d) 1.25×10^{-7}
(e) 2.004×10^{12}

6. Apply the methods of the previous problems to compute the following products by hand. Write your answers in scientific notation.
(a) 1.6×10^{11} times 2.3×10^5
(b) $(1.05 \times 10^{-3})^2$
(c) 1.25×10^{-7} times 2.004×10^{12}

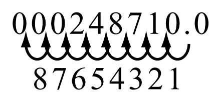
Hints

Hint to problem 4. Be sure to explain the effect of removing decimals (or zeros) from a number and the effect of placing a decimal point a specified number of places to the left (or zeros to the right) of the ones digit of an integer. For example, how do you *mathematically* change 2.38 to 238 and change 0.001045 to 1045?

Hint to problem 5. In part (a), you want to rewrite 1.6×10^{11} as 16 times a power of 10. What do you need to multiply 1.6 by to get the integer 16? What should you then divide 10^{11} by in order to not change the value of the product?

Solutions

1. *Answers may vary.* Since 2 decimal places were removed from 2.38 to get 238 and 6 decimal places were removed to get from 0.001045 to 1045 (that is, $2.38 = 238 \times 10^{-2}$ and $0.001045 = 1045 \times 10^{-6}$), we need to put 2+6, or 8, decimal places back. Counting 8 decimal places in from the ones digit of 248,721 (as in the figure below) gives us 0.0024871, or 2.4871×10^{-2}.



2. Again, answers may vary, but most answers will probably be similar to the following. When given two decimal numbers to multiply together, ignore the decimals and multiply the integers which result. Now, count the number of decimal places you ignored and put the decimal back by counting that many places from the right of the ones digit toward the left, tacking on as many zeros as necessary to the left end of the number in order to place the decimal point.

3. Answers may vary. Most students usually say, "you put the same number of zeros back as you took out in the first place." In this case, then, since a total of 8 zeros were removed (5 from 14,200,000 and 3 from 2,305,000), 8 zeros need to be tacked one to the right of the ones digit of 327,310 to get the final answer of 32,731,000,000,000.

4. First consider the *ignore the decimal* method:
When you ignore 2 decimal places, as you did when you went from 2.38 to 238, you are multiplying the original number by 100, or 10^2. When you ignore the decimal in .001045 in order to get 1045, you are actually multiplying .001045 by 1,000,000, or 10^6. That is,

$$\begin{aligned}1045 \times 238 &= (0.001045 \times 10^6) \times (2.38 \times 10^2)\\ &= (0.001045 \times 2.38) \times (10^6 \times 10^2)\\ &= (0.001045 \times 2.38) \times 10^8\\ &= (2.38 \times 0.001045) \times 10^8.\end{aligned}$$

But then $2.38 \times 0.001045 = \frac{1045 \times 238}{10^8} = (1045 \times 238) \times$

10^{-8}, which is the result of multiplying 238 and 1045, then shifting the decimal point 8 places to the left.

In general, when computing $a \times b$, if A is the result of moving the decimal point in a a total of n places to the right and B is the result of moving the decimal point in b a total of m places to the right, then

Teacher's note: For students that don't have experience with manipulation of symbols involving variables, the general solution might be beyond their reach for now. However, with students with the appropriate background, you might want to insist on the general solution or go over it with them.

$$\begin{aligned} A \times B &= (a \times 10^n) \times (b \times 10^m) \\ &= (a \times b) \times (10^n \times 10^m) \\ &= (a \times b) \times 10^{n+m}. \end{aligned}$$

That is, $a \times b = \frac{A \times B}{10^{n+m}} = A \times B \times 10^{-(n+m)}$, which is the number you get if you take the integer $A \times B$ and slide the decimal point to the left a total of $n+m$ places.

Now, on to the *ignore the zeros* method:
When you go from 14,200,000 to 142, you are dividing by 10^5 (or 100,000, a 1 followed by 5 zeros). When you replace 2,305,000 with 2,305, you are dividing by 10^3. Therefore,

$$\begin{aligned} 2305 \times 142 &= \frac{2305000}{10^3} \times \frac{14200000}{10^5} \\ &= \frac{2305000 \times 14200000}{10^3 \times 10^5} \\ &= \frac{14200000 \times 2305000}{10^8}, \end{aligned}$$

so $14200000 \times 2305000 = (2305 \times 142) \times 10^8$, which you get by tacking on 8 zeros to the right of the ones digit of 2305×142.
In general, when computing $a \times b$, if A is the result of removing n zeros from the right of the ones digit of a and B is the result of removing m zeros from the right of the ones digit of b, then

$$\begin{aligned} A \times B &= \frac{a}{10^n} \times \frac{b}{10^m} \\ &= \frac{a \times b}{10^n \times 10^m} \\ &= \frac{a \times b}{10^{n+m}}. \end{aligned}$$

That is, $a \times b = (A \times B) \times 10^{n+m}$, which is the number you get if you take the integer $A \times B$ and tack on a total of $n+m$ zeros to the right of its ones digit.

5. (a) To get from 1.6 to 16, we have to multiply 1.6 by 10. We now have to divide 16×10^{11} by 10 in to keep from changing the product. That is, we need to decrease the power of 10 by 1, so $1.6 \times 10^{11} = 16 \times 10^{10}$.

Alternate solution: Since $16 = 1.6 \times 10$, we know $1.6 = \frac{16}{10}$, so $1.6 \times 10^{11} = \frac{16}{10} \times 10^{11}$, which equals 16×10^{10}.

(b) If we multiply the left number by 10, we must divide the right number by 10, so $2.3 \times 10^5 = 23 \times 10^4$.

$2.3 \times 10^5 = \frac{23}{10} \times 10^5 = 23 \times 10^4$.

(c) Since $105 = 1.05 \times 100$, we need to divide by 100 (which is 10^2), so $1.05 \times 10^{-3} = 105 \times 10^{-5}$.

$1.05 \times 10^{-3} = \frac{105}{10^2} \times 10^{-3} = 105 \times 10^{-5}$.

(d) Since $125 = 1.25 \times 10^2$, we see that $1.25 \times 10^{-7} = 125 \times 10^{-9}$.

$1.55 \times 10^{-7} = \frac{125}{10^2} \times 10^{-7} = 125 \times 10^{-9}$.

(e) Since $2004 = 2.004 \times 10^3$, we see that $2.004 \times 10^{12} = 2004 \times 10^9$.

$2.004 \times 10^{12} = \frac{2004}{10^3} \times 10^{12} = 2004 \times 10^9$.

6. (a) We know from problem 5 that $1.6 \times 10^{11} = 16 \times 10^{10}$ and $2.3 \times 10^5 = 23 \times 10^4$. Since $16 \times 23 = 368$,

$$\begin{aligned}(1.6 \times 10^{11}) \times (2.3 \times 10^5) &= (16 \times 10^{10}) \times (23 \times 10^4) \\ &= (16 \times 23) \times (10^{10} \times 10^4) \\ &= 368 \times 10^{14}.\end{aligned}$$

Now, $368 = 3.68 \times 10^2$, so $368 \times 10^{14} = 3.68 \times 10^{14+2}$, and the answer is 3.68×10^{16}.

(b) Since $105^2 = 11025$ and $1.05 \times 10^{-3} = 105 \times 10^{-5}$,

$$\begin{aligned}(1.05 \times 10^{-3})^2 &= (1.05 \times 10^{-3}) \times (1.05 \times 10^{-3}) \\ &= (105 \times 10^{-5}) \times (105 \times 10^{-5}) \\ &= (105)^2 \times (10^{-5} \times 10^{-5}) \\ &= 11025 \times 10^{-10} \\ &= 1.1025 \times 10^{-10+4} \\ &= 1.1025 \times 10^{-6}.\end{aligned}$$

(c) In problem 5 you showed that $1.25 \times 10^{-7} = 125 \times 10^{-9}$ and $2.004 \times 10^{12} = 2004 \times 10^9$. Also, $2004 \times 125 = 250,500$, so

$$\begin{aligned}(1.25 \times 10^{-7}) \times (2.004 \times 10^{12}) &= (125 \times 10^{-9}) \times (2004 \times 10^9) \\ &= (125 \times 2004) \times (10^{-9} \times 10^9) \\ &= 250500 \times 10^0 \\ &= 2.505 \times 10^{0+5} \\ &= 2.505 \times 10^5.\end{aligned}$$