Topic Study: Functions – Represent, Analyze, Generalize

Background:

This Topic Study includes sections dealing with mathematical context, pedagogy, essential questions to elicit student thinking and prompt mathematical understanding, potential errors and misconceptions associated with the topic, technology integration considerations, and possible topics for collegial dialogue.

The tool that is the primary focus during this study, ***Algebra Arrows*** (Freudenthal Institute, 2004), is designed to generate functions. This applet also allows the user to show multiple representations of the function including as an expression, as a table, and as a graph. It also is designed to evaluate the expression if so designated. It is not the only tool appropriate for this topic, however, as many other tools support students in developing and deepening their understanding of functions. Some additional interactive tools are ***Introduction to Functions, Linear Functions, Using Tables, Rules, and Graphs.*** (Explorelearning, 2005), ***Whole Number Cruncher, Function Machine, Linear Function Machine*** (Shodor Education Foundation, 1997-2005), and ***Algebra Trees*** (Freudenthal Institute, 2004) to name a few.

Context

A function is a rule to explain a relationship between an input and an output. It uniquely defines the value of one variable in terms of the other. It is a special kind of relation and can be shown as a set of ordered pairs in which every value of x has a unique value of y. Students can begin studying the concept of a function as early as third grade and continue the study of functions through and beyond high school. The study of functions should be embedded in real life contexts whenever possible and should include fluency between and among multiple representations of that function. These representations can include an expression of the rule in words, the functional relationship recorded in a table, a list of ordered pairs, a graphical representation of the function, or an equation to represent the function.

<u>Possible Representations:</u>

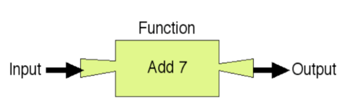
Rule in Words

Add 7 to the Value

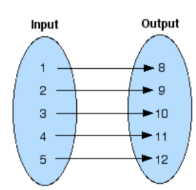
Equation

$y = x + 7$

Function Machine



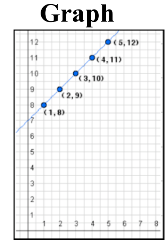
Arrow Mapping/Diagram



Table

Input	Output
1	8
2	9
3	10
4	11
5	12

Graph



Research has shown that when students examine the ordered pairs of function that are listed in a table, they will first look at patterns vertically in the table – for instance, in a table of values for the function $y = 2x+1$, they will tend to notice first that "the numbers in the y-column go up by 2 each time." Most students need various experiences with patterns of values listed in tables in order to begin to look for patterns across the two columns and to relate the two sets of numbers – to then say, "You double the first number then add 1." The ***Algebra Arrows*** tool helps focus students on both "vertical" patterns in the table as well as "horizontal" patterns across the two columns. By showing the different mathematical steps of the function with each arrow, the pattern across the columns is made explicit. With a tool such as ***Algebra Arrows***, students can begin with less complex real life relationships then build up to more complex scenarios.

Once students have had experiences creating functions, they can begin to explore inverse functions. Understanding inverse functions is a sophisticated mathematical skill for many students and not trivial, even though it may seem apparent that by simply using inverse functions in the reverse order, students can easily build an inverse function. Students need many experiences thinking about how to both "undo" each operation and to perform the "undone" operations in reverse order. Tools such as these make the "doing" and "un-doing" much more explicit.

Pedagogical Considerations

This particular tool can be used for many different activities related to pattern-seeking, graphing, understanding functions and comparing types of functions. Because there are so many key mathematical ideas that can be addressed, it is important to clarify for yourself and for students which mathematical goals you are focusing on in any given lesson with this tool, and focus your questions accordingly.

Good, thought-provoking discussion questions are key to accompanying this tool – without appropriate follow-up discussion questions, students will do little more than have fun experimenting with an interesting online tool.

You might decide to focus your discussion questions on any of the following topics:

- patterns that appear in both the table and the graph;
- the relationship between the table and graph;
- interpreting the graph and understanding what it is representing;
- why the graph of a given function is a line rather than a set of discrete points.

Essential Questions

To probe for understanding:

- When you select a value in the input column, why does another value in the output column also get highlighted?
- Why do the red lines shown on the graph move each time you select a different point from the input column?
- What happens to the output values when you change the order of the steps in the function? Why do you think the order of the steps matters?

To prompt mathematical reasoning:

- What patterns do you notice going down the table?
- What patterns do you notice between the input column and the output column?
- What patterns do you notice on the graph?
- How are patterns in the graph connected to patterns you see in the table?
- If you were describing to a friend how to take a function and create the "reverse" (inverse) function, what would you tell them to be sure to do? What rules or advice could you give that would work for any function they used?

Potential Errors and Misconceptions

Error: Students may not use the input or output boxes in their arrow string.

Related misconception: Students may be leaving out the input or output boxes for several reasons. One possibility is that they do not understand the Arrow model; while they may understand that they are creating an expression into which they can plug various numbers and see what results, they may not understand what the purpose of the "white" or empty boxes are in the Arrow model.

Another possibility is that they do not understand the concept of function as a collection of corresponding input and output numbers. They may see that they need to try to make an expression but not understand that they are searching for many sets of corresponding input and output numbers.

Caution: Students may change the axes on the corresponding graphs in the applet so that the scales do not match up with each other. While this is not an error, it is important for students to explore why the graphs look different if you change the scale.

Related misconception: It is possible to resize the graph to change the intervals between each line. However, students may not understand the significance of changing one set of axes without changing the other to correspond.

While it is technically ok to have the axes be at different scales, students can easily be misled about what the graph actually looks like. For instance, a student who has resized the y-axis only for the equation $y = 3x + 1$, in order to see more points, may think that the line is much less steep than it actually is.

This may be due to a student's lack of understanding of graphs as a representation, and in particular, the importance of selecting appropriate scales for the axes. Students are likely not to understand that by resizing the axes, their steeper (or less steep) graph represents the same function as another student's graph that has not been resized. Watch for students who interpret the graphs very concretely in this way; they may be showing evidence that they are struggling with the abstract nature of graphs as a representation.

It is also critical to have students focus in on the scale when comparing the graphs of two different equations if the equations are graphed separately rather than on one graph.

Error: When asked to find the input number given the output number, students may use inverse operations but in the same order as they were in the original problem. For example, if a pattern results from the expression $2x + 1$, they will calculate the inverse pattern as $x \div 2 - 1$, instead of $([2x+1]-1) \div 2$.

Related misconception: The concept of "undoing" is a difficult one for students who are first encountering it. Most middle grades students have some understanding of an inverse operation, but often are not yet clear on how to apply those inverse operations in more than one step. Students may need to fall back on describing the inverse set of steps verbally and testing various output numbers to see if the result is the corresponding input number. It is easy to show "undoing" with algebra arrows.

Technology Integration

In order to see the table of values there must be a variable in the input box. This is a feature that makes it easy to show the difference between a “value” and a “variable.” In order to see the graph, you must connect the arrow of the output to the graph. If you wish to show the difference between two functions using the same input, you must connect the same input to each function.

As students build a function with the algebra arrows, they can observe how the steps of the function occur in order. This applet lets students easily change the order of mathematical steps in a function and see how the resulting y-values change. It is important mathematically to let students experiment with changing around the order of the steps in a function in order to see that this order of the steps does matter. This kind of exploration lays the foundation for exploring the inverse of a function – in which they try to build a function that begins with the output values and results in the input values.

Understanding inverse functions is a sophisticated mathematical skill for many students and not trivial, even though it may seem apparent that by simply using inverse functions in the reverse order, students can easily build an inverse function. Students need many experiences thinking about how to both “undo” each operation and to perform the “undone” operations in reverse order. This tool helps make explicit both these aspects of inverse functions in a visual way.

When the table is shown in the applet, there are up- and down-arrows for each axis that allow the user to change the scale of the axes. Be cautious about how students use these arrows as it is easy for students to set the axes to different scales and distort the shape of the graph without realizing they have done so. For linear functions in particular, if the scale is different on the axes, the slope will remain numerically accurate, but will visually appear tilted at different angles, depending on where the axes are set.

This tool can also provide a strong basis for an activity that explores slope as a ratio. As you select different values in the table view, the corresponding points are highlighted on the graph and red lines outline the rise and run for each point. As students count the squares for the rise and run of the slope and record the results, a pattern of proportional numbers will become apparent. This can lay the basis for a good discussion about the relationship of the pattern of values in the table and the pattern of rise- and run-values in the graph.

Professional Community: Collegial Dialogue

Ask colleagues:

- What are some ways in which you've chosen to use this tool? What math ideas did you focus on?
- What kinds of discussion questions did you find most useful in drawing out students' thinking?

Reflection questions:

- What do you think the tool allows your students to do that they could not do as easily on paper?
- What do you see as the merits of using paper-and-pencil and of using the online tool? How can they best work together?

Related Readings:

Representation As a Vehicle for Solving and Communicating, Ronald Preston & Amanda Garner
MTMS, September 2003, Volume 9, Issue 1, Page 38

Reasoning About Linear Equations, A. Susan Gay & Charlotte J. Keith
MTMS, November 2002, Volume 8, Issue 3, Page 146

Tools Referenced:

Explorelearning. (2005).

Freudenthal Institute. (2004). Algebra Arrows.

Shodor Education Foundation. (1997-2005). Project Interactivate Applets