Building Standard Algorithms

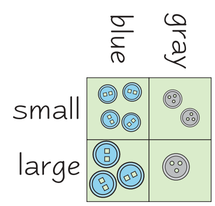
From Buttons to Algebra

excerpted from Teacher's Guide introductory materials

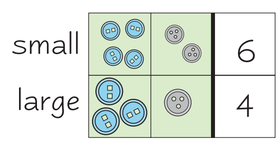
To build skill with understanding, *Math Workshop* uses children's own natural ideas about sorting objects as a foundation for the standard algorithms of addition, subtraction, and multiplication.

Learning to add by sorting

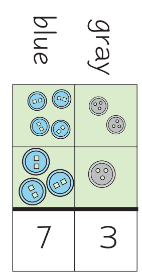
Imagine a young child with a small pile of buttons—some large, some small, some gray, and some blue. Most children notice these characteristics readily and seem naturally drawn to sorting. Some children sort in only one way, separating gray from blue, or large from small. Others make four piles, arranged in no particular order. As teachers, we help children refine and extend what children do naturally by adding *organization* and *language* to their natural inclination to classify. 'Gray' and 'blue' are not just any characteristics. They are the same kind; they are both *colors*. Similarly, 'large' and 'small' are both *sizes*. Classifying characteristics of buttons (for example, classifying 'blue' and 'gray' as *colors*), rather than classifying the buttons themselves,is a new level of abstraction that children begin developing in pre-school and continue through kindergarten. This intellectual growth takes place even without formal teaching, but good examples and activities can hone, nurture, and extend it. One way to help children reduce four characteristics (gray, blue, large, and small) into two kinds of characteristics (color, size) is to have them arrange their piles according to the two attributes. This labeled arrangement is a concrete representation of what we've learned about the buttons through sorting them, something like a hybrid of a table and a data graph.



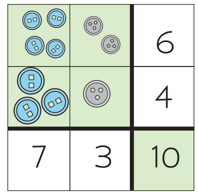
The method of sorting, the very categories that children choose, suggests other ways of summarizing the data. Children see these alternatives quite naturally. How many small buttons are there? How many large ones? The answers to these new questions can be summarized directly in the table, like this.



Of course, we can equally well ask how many buttons are blue, and how many are gray. Again, the answers can be summarized directly in the table. That the total of all blue and gray buttons (7 + 3) should be the same as the total of all large and small buttons (6 + 4) makes complete sense, because both of these combinations (blue and gray, or large and small) are ways of referring to all the buttons.



There are 10 buttons in all, no matter how we might sort them into smaller categories.



If we substitute numbers for the original objects, the situation remains the same.

4	2	
3	1	

We can total across to get 6 and 4 (the numbers of small and large buttons), and we can total down to get 7 and 3 (the blue and gray buttons). These two pairs of numbers have the same sum. The way the numbers fit neatly together is reminiscent of a crossword puzzle, so *Math Workshop* calls this format the Cross Number Puzzle.

4	2	6
3	1	4
7	3	10

From their natural understanding of counting objects, children have proof that any set of starting numbers will lead to just as neat a fit: the sum of the right-side white boxes must be the same as the sum of the bottom white boxes.

The important idea here is that addition works *in any order* and *with any grouping*. In the puzzles above, children may group the numbers (or buttons) by rows as 4 + 2 and 3 + 1 and then add those results, or they may group the numbers by columns as 4 + 3 and 2 + 1 and then add those results. Either way, the total must come out the same.

Learning facts through puzzle solving

Before looking deeper into the serious side—the usefulness of this idea to children's learning, *with understanding*, the addition and multiplication algorithms, and more—let's look at the more playful side. Play is one of the major ways that children get practice to develop genuine skill, whether that skill is a physical one like becoming proficient at bicycle riding or basketball or an intellectual one like becoming fluent with number facts.

Even realizing that they may add numbers in any order doesn't help children until they can find some order that lets them calculate with less effort. This requires that some facts come absolutely automatically, and learning number facts takes repetition—lots of it. But sheer volume of practice is not enough. Repetition alone quickly becomes boring, and attention wanders. Children (and adults) can repeat and repeat and still not learn, if their attention is not on what they are doing.

One way to keep a student's attention alive while gaining necessary practice is to embed the practice in a puzzle. Here's a variation in which the student doesn't start with the four green boxes filled in, but must instead figure out what numbers might have been in those boxes from the available clues.

5		8
	9	21

This is a *real* puzzle. Organization and strategy are needed to figure out what numbers go in the empty boxes. Even knowing where to start can be a challenge. Whatever choices a child makes will lead to a great deal of addition and subtraction practice. Puzzles also help to develop students' problem solving strategies.

How the learning of facts evolves into the learning of methods

Over the course of doing many Cross Number Puzzles, students gain proficiency in the facts they need for addition and subtraction, and they learn when each operation is required. They also gain experience with, and begin to develop a sense of, the "any order, any grouping" principle of addition, which is the underpinning of the standard algorithm for multi-digit addition.

Consider, for example, one of the common ways to teach children how to add two numbers like 25 and 38. First write the numbers, carefully lining up the columns, and then add the 5 and 8 to get 13. In a separate step, add the 20 and 30 (sometimes confusingly called "two" and "three") to get 50. The most common algorithm combines

$$\begin{array}{r} {}^{1} \\ 25 \\ +\;38 \\ \hline 63 \end{array}$$

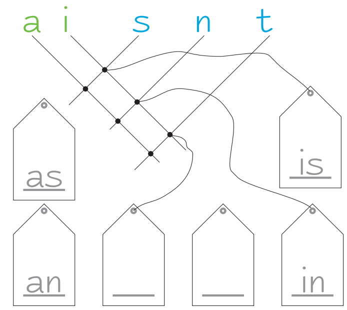
those two results by carrying 10 from 13 to the 50. The numbers 13 and 50 don't actually appear explicitly on the paper, but they do appear in the student's mind, and even out loud if the student is still learning.

20	5	25
30	8	38
50	13	63

Students who have worked with the logic of Cross Number Puzzles will find every step in this algorithm quite familiar. In this example, the original 25 and 38 appear at the right, just as they might in a standard vertical-addition problem. The 25 and 38 are also shown as the results of adding 20 to 5 and 30 to 8. The bottom row shows the sums 5 + 8 and 20 + 30. The most important idea is that 50 + 13 gives the same result as the computation we wanted to perform, 25 + 38, so we can perform that simpler computation instead. This is exactly the method that the standard addition algorithm uses. People often think that they add 25 and 38, but really, they form simpler computations that they know will give the same answer as 25 + 38.

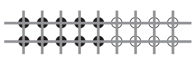
It takes just one more small step for the ideas that students develop from the Cross Number Puzzle to become the foundation for the multiplication algorithm.

An image of multiplication, and a method for multiplying

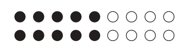


Two streets and three avenues cross. Each intersection is named after the two roads that meet there. Which intersection is named *as*? What's the name of the intersection of A-street and N-avenue?

This playful activity, which uses reading and spelling as context, introduces many mathematical ideas. The number of intersections is the number of streets times the number of avenues. When labeling the intersections, students practice the same skills that they need as they learn to read information from tables, and later when they learn to read coordinate graphs. But there's even more buried in this idea.

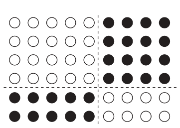


Let's look only at the streets and avenues, ignoring their names, in a slightly larger "town." This tiny little town has just two avenues going east and west, and nine streets going north and south. How many intersections are there?

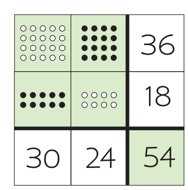


If we now ignore even the roads and look only at the intersections, we see them as arrays of dots, arranged in rows and columns. An arrangement of two rows, each containing nine dots, will contain eighteen dots: 2 × 9 = 18. Coloring the dots in five of the columns doesn't change the total number, but it does suggest that 2 × 9 describes the same number as 2 × (5 + 4). The parentheses make this idea look complicated—and when we teach children about this, it is often part of a hard-to-remember set of rules (order of operations)—but the idea is really a very intuitive one. Given a bowl of mixed fruits (say, five blueberries, and four limes) and the instructions to double the amount of fruit, a child might naturally be inclined to double each part, making ten blackberries and eight limes.

The same idea applies to numbers just as it applies to fruit bowls, and when we teach arithmetic, we want to build upon the child's experiences and natural understandings. To double the whole thing, we double each of its parts. If we don't know how to double some number (in this case, 9), we can take that number apart, double the parts separately, and then add the results.



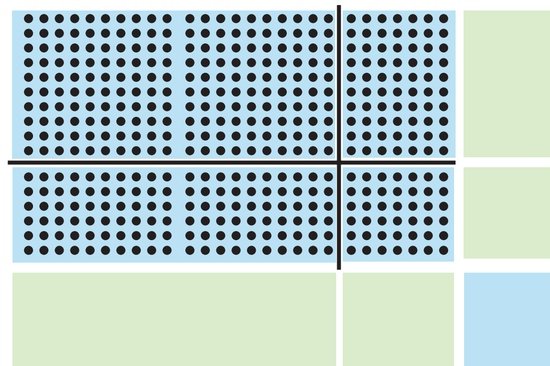
This flag-like arrangement shows six rows of nine dots each. If we know what 9 × 6 is, we can easily tell how many dots in all. But suppose we don't know that fact. The nine columns are grouped as 5 + 4, and the six rows are grouped as 4 + 2. Whatever 9 × 6 is, we can see that it is the same as the sum of four other products: 5 × 4 + 5 × 2 + 4 × 4 + 4 × 2.



When students take algebra, they learn to multiply $(a + b) \times (c + d)$. Your students will already know the idea behind the rules. They will have learned it from picturing a 9 × 6 array of dots grouped as (5 + 4) × (4 + 2), and then adding each separate region of the array. The adding, of course, can go in any order at all. Two special ways suggested by the array itself—adding across and then down, or adding down and then across—remind us of Cross Number Puzzles.

Knowing the answers to some multiplication problems (in this case, the multiplication of small numbers by 4 or 5) allows people to figure out the answers to other multiplication problems (in this case, multiplication by 9) by using a little bit of addition. The 4- and 5-times tables were used as the "basic facts" for multiplication by 9. With a small store of memorized answers—just the answers to multiplications up to 9 × 9—this idea can be extended to solve any multiplication problem.

The problem 16 × 27 can be pictured as 16 rows each with 27 dots. In this picture, the vertical line cuts each row so that 20 dots are on one side (with a space to help us see the 20 as two groups of 10), and 7 on the other. The horizontal line groups the rows, also by keeping tens (in this case, only one of them) separate from the "leftovers." The horizontal and vertical lines separate the original problem into four new problems, but they are easier ones. At the top left, we see 200 dots; top right, 70; bottom left, 60 × 2; and bottom right, 6 × 7. When we summarize these results as a Cross Number Puzzle, we see the skeleton of the standard algorithm for multiplying two-digit numbers. Compare the numbers you see in the Cross Number Puzzle with the numbers in the two computations to its right.



200	70	270
120	42	162
320	112	432

$$\begin{array}{r} 16 \\ \times 27 \\ \hline 112 \\ 320 \\ \hline 432 \end{array} \qquad \begin{array}{r} 27 \\ \times 16 \\ \hline 162 \\ 270 \\ \hline 432 \end{array}$$

Students need less time to learn the standard algorithm for multiplication, and they develop greater understanding of why it works, if they build these rules on a firm foundation of ideas about arrays, objects, and the summary tables that we call Cross Number Puzzles. These ideas apply to the learning of subtraction and division, as well, and the payoff continues beyond arithmetic and into algebra.

The algebraic connection

Here's a great trick that provides yet more arithmetic practice. Write down two addition sentences, lining them up carefully, one below the other, like this:

$$\begin{array}{l} 2 + 4 = 6 \\ 3 + 6 = 9 \end{array}$$

Now, below these two, write the sum of each column of numbers, like this:

$$\begin{array}{ccccc} 2 & + & 4 & = & 6 \\ 3 & + & 6 & = & 9 \\ 5 & & 10 & & 15 \end{array}$$

The resulting three numbers *also* make a true addition sentence.

$$\begin{array}{ccccc} 2 & + & 4 & = & 6 \\ 3 & + & 6 & = & 9 \\ 5 & + & 10 & = & 15 \end{array}$$

Is this always true? Yes! Why? Remember the buttons! Starting with the numbers 2, 4, 3, and 6, we can create a Cross Number Puzzle. If we add across, we get the original number sentences: 2 + 4 = 6 and 3 + 6 = 9. If we add down, we get the numbers 5 and 10. But we can then add those sums (across) to produce a new

2	4
3	6

number sentence: 5 + 10 = 15. The Cross Number Puzzle logic—the logic of the buttons—tells us that the 15 is exactly what we'd get by adding the results of the previous two additions, 6 + 9.

At some point, students learn to solve "simultaneous equations" by doing the same kind of adding or subtracting of number (and letter) sentences to make new ones. An equation like $5x + 3y = 17$ is just a number sentence like 2 + 4 = 6, or at least it would be if only we knew what x and y were. We know we can add (or subtract) number sentences to make new ones that are true. For this pair of number sentences, subtracting seems best.

$$\begin{array}{rcrcr} 5x & + & 3y & = & 17 \\ 2x & + & 3y & = & 5 \\ 3x & + & 0 & = & 12 \end{array}$$

The first subtraction ($5x - 2x$) gives us $3x$; the second and third subtractions give 0 and 12. The new equation, $3x = 12$, tells us that x must be 4. Life just got much simpler.

All from sorting buttons!