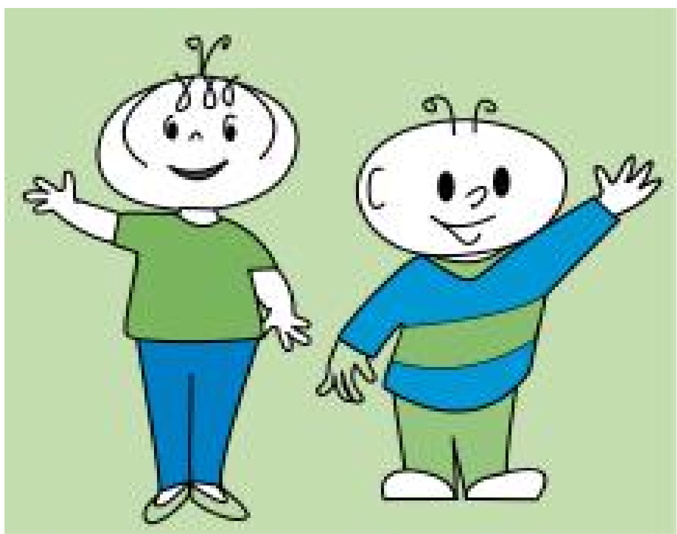
Making Time for Problem Solving and Also Building Basic Skills



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Within EDC, the Division of Mathematics, Learning, and Teaching (MLT) creates curriculum materials, does research in mathematics education, and designs professional development for teachers, administrators and other leaders.

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Math Workshop is a National Science Foundation funded project, producing a K-5 mathematics curriculum that balances deep understanding with purposeful computational practice.

http://www2.edc.org/mathworkshop/

The *Math Workshop* curriculum will be published by Harcourt School Publishers under *a name to be determined later*.

Please give us your feedback!

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- Or write a brief response on the last page of your handout and give it to us today.

Thank you!

Counting: From counting objects to counting ways

	T	L
Blue		L L
Green	T T T	L

0	2
3	1

0	2	2
3	1	4
3	3	6

From sorting by two attributes, students move to counting objects, and then to counting objects in several different ways. They also learn to use the fact that there is more than one way to count objects to check their answer.

$$\begin{array}{r} 2 \\ +\ 4 \\ \hline \end{array}$$

$$3 + 3 = 6$$

Counting: From counting objects to counting ways

8	2	
4	3	

Students complete these charts with or without manipulatives.

	2	5
3	3	
		11

In the charts, different numbers could be missing. The amount of information provided varies. Having some extra information allows for students to use more strategies to complete the puzzle and to verify the answers. In addition, it raises the question: do we have some extra information? Is it possible to erase a number and still be able to complete the chart?

4		
3		
		9

Then, sometimes there is not enough information, and a chart can be completed in more than one way.
In how many ways?
What are these ways?
How can you make sure all the ways are found?

Counting: From counting objects to counting ways

Students solve a variety of problems like:

How many ways can you arrange 8 things in two boxes?

or

How many ways can you complete __ + __ = 8?

and then move to more difficult problems, like

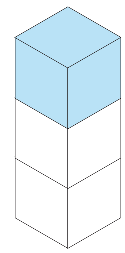
If there are 8 lines, each vertical or horizontal, how many crossing points are there?

In this problem, students utilize their ability to consider different cases systematically. On the way, students practice multiplication and collect data for a new research problem, which they will formulate later:

For a given number of lines, when is the number of crossing points the largest?

Counting: From counting objects to counting ways

How many different block towers?



Rules: 1. One blue block, no more and no less.
2. Enough white blocks to make a given height.

A. How many different towers are one block tall?

Show your towers here.

B. How many different towers are two blocks tall?

Show your towers here.

C. How many different towers are three blocks tall?

Show your towers here.

D. How many different towers are four blocks tall?

Show your towers here.

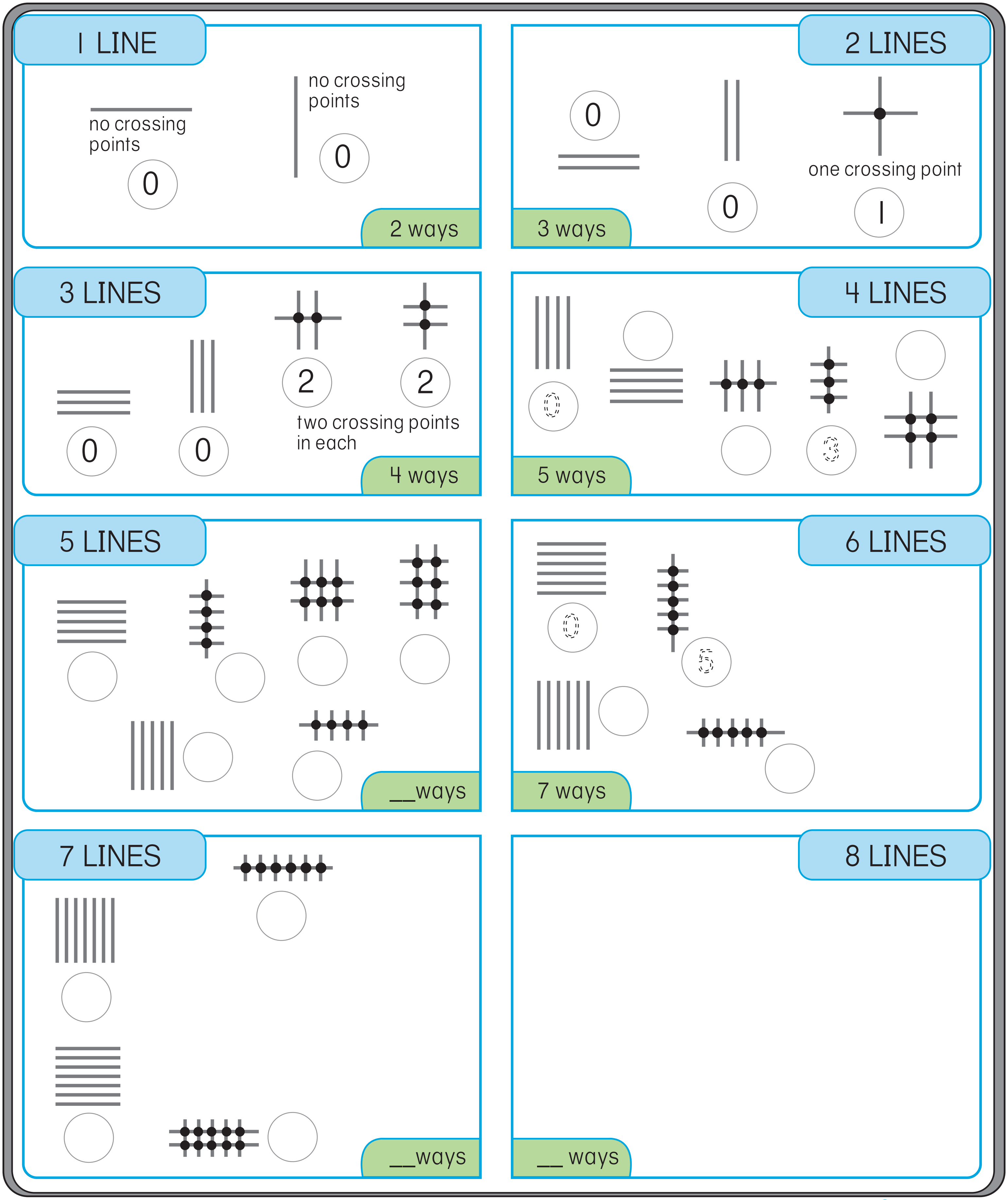
Changing rules:

New rules: 1. Two blue block,s no more and no less.
2. Enough white blocks to make a given height.

How many different block towers?

Counting: from counting objects to counting ways and systematization

Each line is either vertical or horizontal. How many crossing points?



Missing Parts—Addition and Subtraction

You can create engaging puzzles just by erasing unexpected numbers or digits in a standard arithmetic problem. By figuring out what is missing, students develop a deeper understanding of what an operation does and how to undo it. The level of difficulty of the puzzle depends on what you erase, and you can leave more clues than are absolutely necessary if you wish. It's also okay to have a problem with multiple solutions, such as __ + __ = 10.

$2 + \square = 5$ $8 - \square = 3$

$10 + \square = 14$ $\square + 6 = 96$

$\square - 90 = 6$ $40 = 47 - \square$

$53 - \square = 40$ $\square - 70 = 18$

Problems like these help students learn to take addition and subtraction problems apart. They also begin to see two-digit numbers as having a tens part and a ones part. The problems reinforce the connections between addition and subtraction.Students may use a wide variety of strategies to fill in the blanks.

When the original problem involves subtraction, students must take care to figure out the role of each number they can see. Which are parts? Which is the whole? If you vary the problems widely, students will never stop thinking and follow a rote procedure, but will keep their brains switched on! Just a small change is enough to keep students' attention.

$\square 4 + 1\square = 67$ $\square 7 - 5\square = 13$ $38 + 2\square = \square 8$

$25 - \square 3 = 1\square$ $\square 1 + 86 = 9\square$ $7\square - 53 = \square 4$

Here, students use the two-digit structure of the numbers in each problem to figure out the missing digits. By handling tens and ones separately, they are preparing for standard American algorithms.

If you mix addition and subtraction problems on a page, you add to the unpredictability that can help students realize the consistent need to read problems carefully before starting to work.

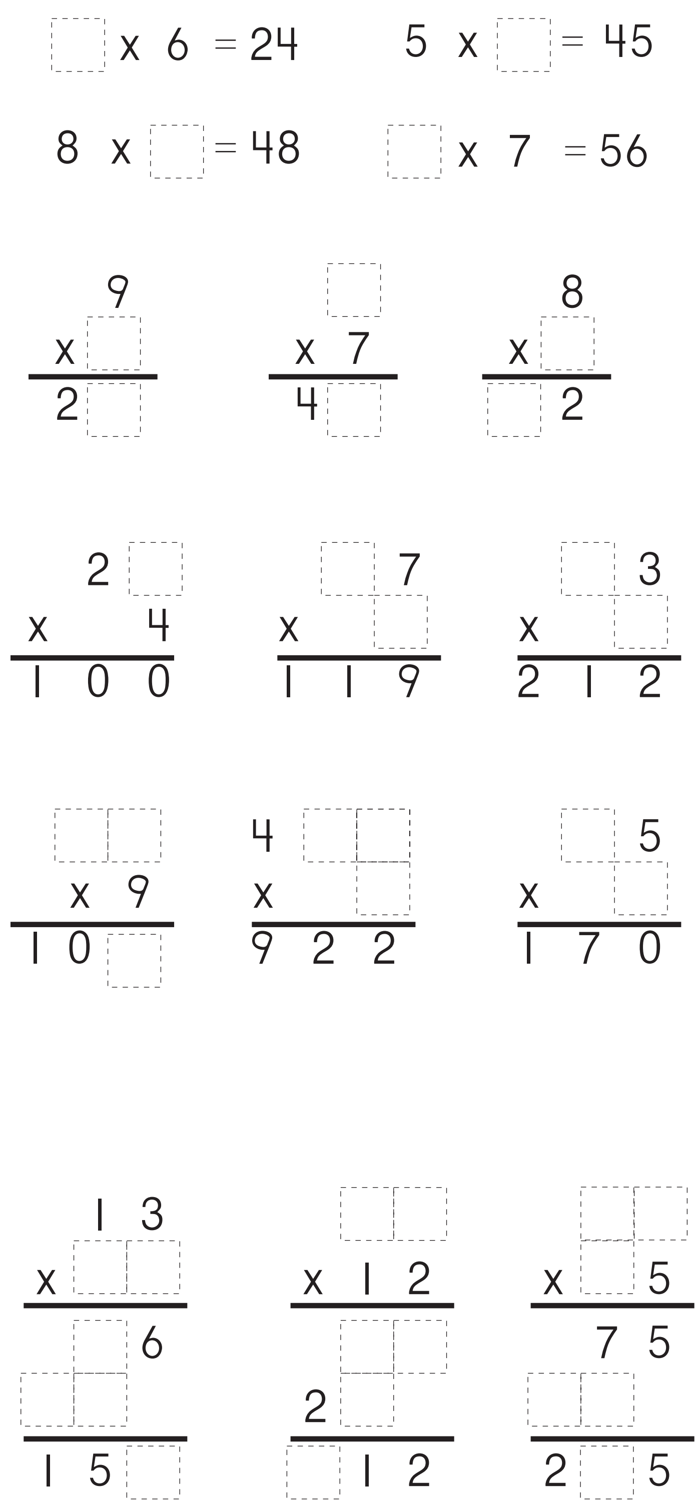
$8\square - 36 = \square 8$ $\square 6 + 29 = 8\square$ $3\square + \square 8 = \square 15$

$73 - 2\square = \square 8$ $27 + \square 6 = 6\square$ $\square 0\square - 73 = 1\square 7$

When you hide digits in problems that involve re-grouping, students can't use a rote procedure to solve them. They must figure out where an "extra" ten comes from and where it belongs. They see the effect of the re-grouping on the rest of the problem.

Students also learn the value of checking their answers. These problems are difficult enough that students want confirmation for themselves that their answer is correct.

Missing Parts—Multiplication



In the early stages of learning multiplication facts, missing factor problems can cause students to review many products of the known factor to find the answer. They can also use estimation skills.

Missing digit problems can also lead students to use estimation. “What multiple of 9 is in the 20s?” This kind of estimation is critical in long division. Problems with more than one possible solution (like the third problem here) challenge students to consider more possibilities.

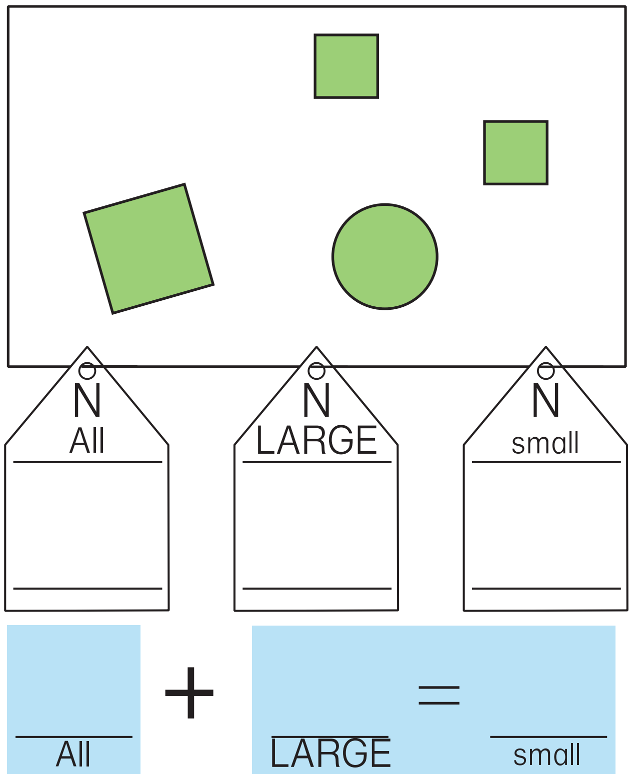
In finding ending digits, students often gain intuition about divisibility. “What kind of number can you multiply by 4 to get an answer that ends in 0?” They run through many single digit facts to find candidates to test. For difficult problems or new types of problems, consider providing more information than is absolutely necessary.

Puzzles like these let students think about approximate size. “I know the other factor has to have 2 digits. The first digit has to be 1, because 9 x 20 is too big (180). The second digit should be small, because 10 □ isn’t close to 180. 11 x 9 = 99. Not big enough. 12 x 9 = 108. That works. 13 x 9 = 130 – 13. Too big—the second digit won’t be 0. (117) In the second example, 400 is a little less than half of 900, so maybe this is something multiplied by 2.

As students work with larger and larger puzzles, they will develop strategies that depend on all the techniques they used to solve the smaller problems. Ask students, “What number did you guess first and why?” In the first problem, they might say something like, “Something times 3 ends in 6. That must be 2 in the ones place of the missing factor.” In the second, a student might fill in a 2 in the second box in the third row by seeing that the final entry in the ones column will be a 2. That leads to 2 possibilities for the ones digit of the missing factor.

There are many ways to attack these problems, so you can’t predict how students will find the solutions. However, you can predict that they will need to develop a deep understanding of the multiplication algorithm to be successful.

Algebra



From the beginning of grade 1, students can be exposed to using letters as labels, as place holders, and even in algebraic expressions, which they can use as additional hints, but also can choose to ignore.

A. Find a Rule

n	7	13	42	5		28			8
n+10	17	23	52		33		16	10	

Algebra

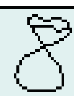
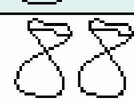
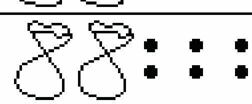
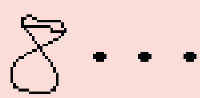
A think-of-a-number trick:
I know your answer!

Think of a number.
- Add 3.
- Double the result.
- Subtract 4.
- Divide the result by 2.
- Subtract the number you first thought of.
- Your answer is 1!

When students are surprised by a trick, they want to repeat it several times. (Think of all the arithmetic practice involved.)

Students use word and pictorial clues and multiple strategies to study a think-of-a-number trick; meanwhile, algebraic notation is evolving.

A think-of-a-number trick: what was the number?

Think of a number.		5			
Double it.					
Add 6.					
Divide by 2. What did you get?			7	3	20

Algebra

Sue made up a trick. Reconstruct the records.

Shorthand	Anne	Ben	Cindy	Dan
x				8
		17		18
x + 4	7	11	15	
	4		4	

Note the growing level of difficulty and increasing use of notation.

Can you reconstruct these records?

Shorthand	A	B	C	D
x				7
		0		
2x + 20		20	26	34
	26	20		27
			5	
2				

Algebra

The familiar format is used to pose another problem.

(Try some examples of your own.)

Words	Shorthand	Ben	Lee	A	B	C	D
Think of a number	n	3	3				
Multiply your number by itself	n x n	9					
Subtract 1 from the product	(n x n) - 1	8					

Add 1 to your number	n + 1	4					
Subtract 1 from your number	n - 1	2					
Multiply your results together	(n + 1)x(n - 1)	8					

For any number, is it true that $(n \times n) - 1 = (n + 1) \times (n - 1)$?

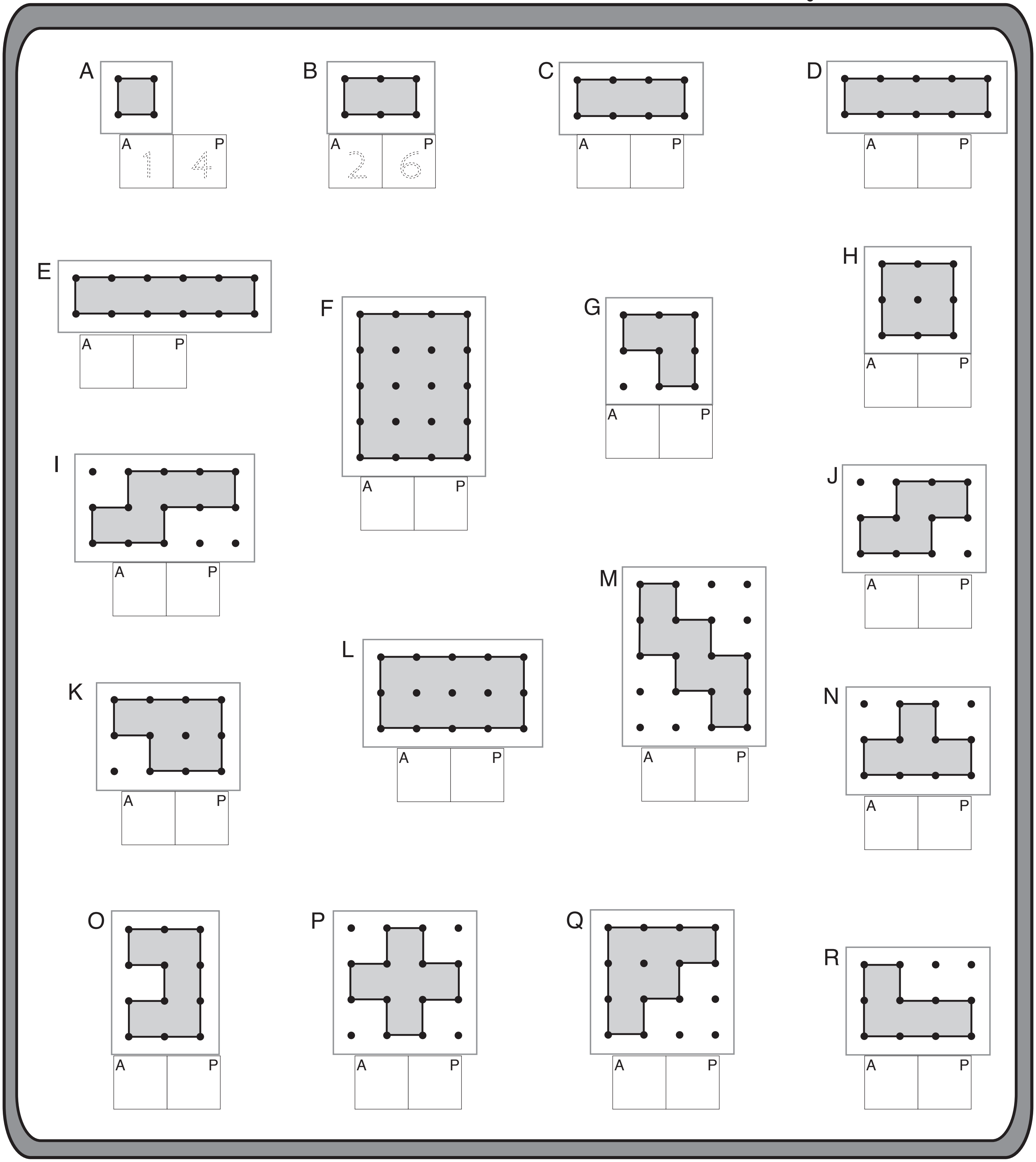
How to make sure it is always true?

How can we use this fact?

Name:______________________ Date:______________ Time:___________

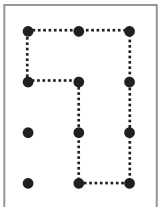
A ... on this page is **AREA**. Here is one unit of area:

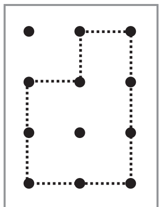
P ... on this page is **PERIMETER**. Here is one unit of perimeter: or

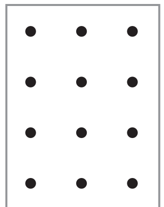


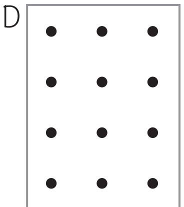
Name:____________________ Date:______________ Time:___________

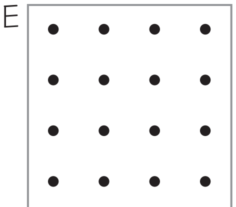
Using only horizontal and vertical lines, draw sketches of the **areas** and **perimeters** shown in the boxes marked **A** and **P**.

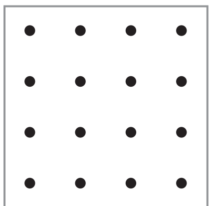
A

A 4 P 10

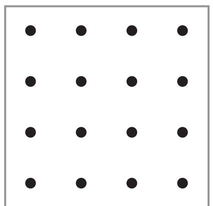
B

A 5 P 10

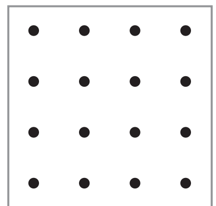
C

A 6 P 10

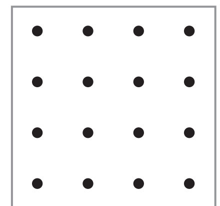
D

A 5 P 12

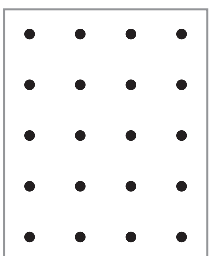
E

A 9 P 12

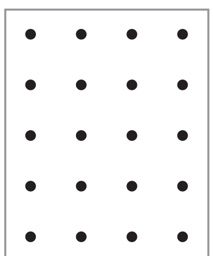
F

A 8 P 12

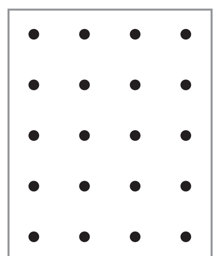
G

A 6 P 12

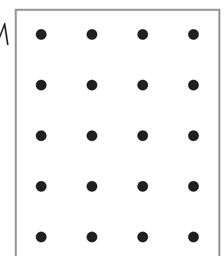
H

A 7 P 12

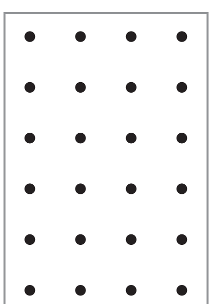
I

A 8 P 14

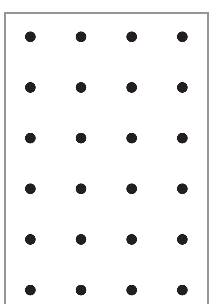
J

A 11 P 14

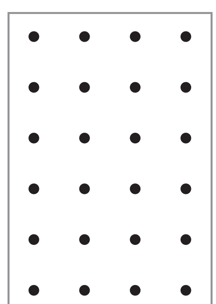
K

A 12 P 14

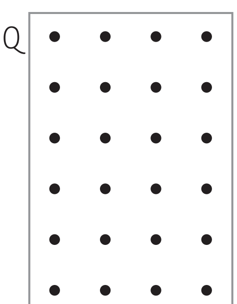
L

A 9 P 14

M

A 10 P 16

N

A 7 P 16

O

A 15 P 16

P

A 10 P 16

Q

A 11 P 24

Name:________________________ Date:______________ Time:___________

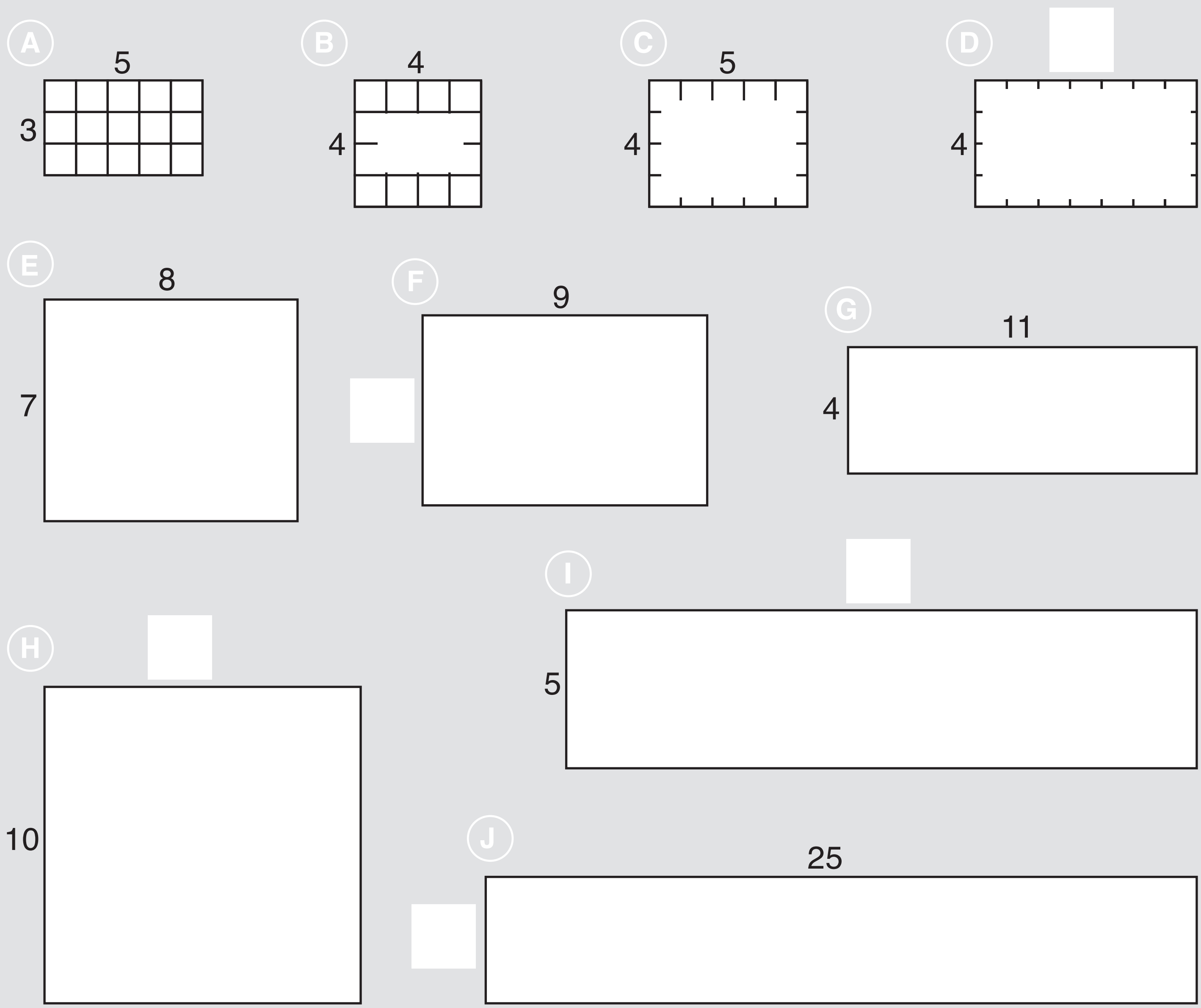
A ... on this page is the **AREA** measured in square units that look like this: □

P ... on this page is the **PERIMETER** measured units that look like this: — or |

Draw a shape with the correct area and the smallest possible perimeter.

	A	P
A	4	8
B	5	
C	6	
D	7	
E	8	
F	9	
G	10	
H	11	
I	12	
J	13	
K	14	
L	15	
M	16	
N	17	
O	18	
P	19	
Q	20	

Name:______________________________ Date:____________________



	A	B	C	D	E	F	G	H	I	J
LENGTH	5	4	5		8	9	11			25
WIDTH	3	4	4	4	7		4	10	5	
AREA	15			28		54		100	100	100
PERIMETER	16									

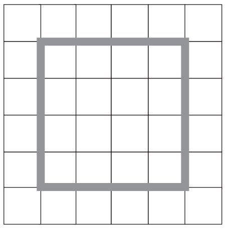
Name:______________________________ Date:_________________________

Find the area and perimeter of each figure.

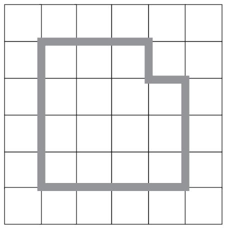
☐ = 1 square unit

— or | = 1 unit

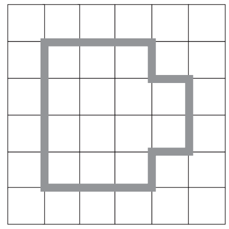
A



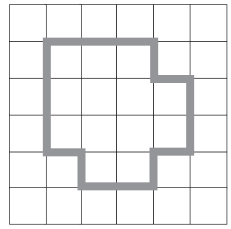
B



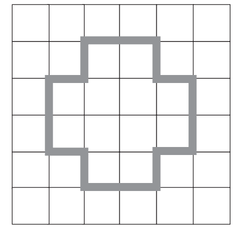
C



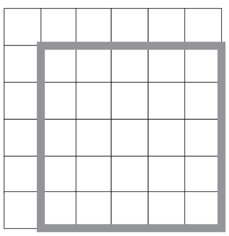
D



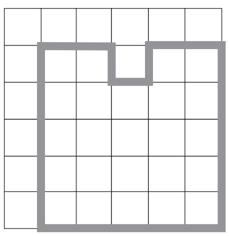
E



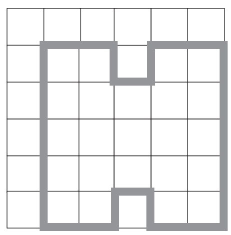
F



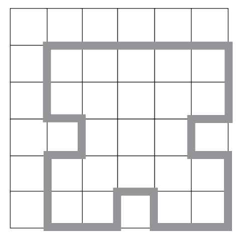
G



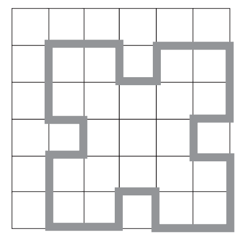
H



I



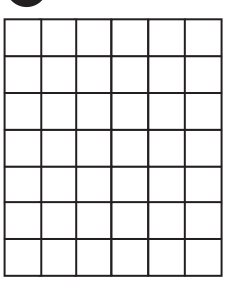
J



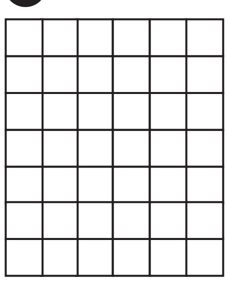
	A	B	C	D	E	F	G	H	I	J	K	L	M	N
AREA	16										12	11	10	9
PERIMETER	16										14	16	18	18

Draw a shape with the area and perimeter given in the chart above.
Draw only on the lines.

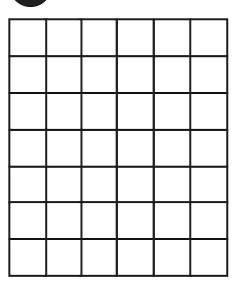
K



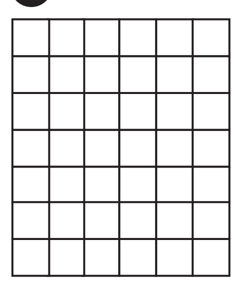
L



M



N



Some notes on Area & Perimeter

Here are some questions and extensions that can help bring out problem solving in the five Area & Perimeter pages in this handout.

Page 1:

Problems A through E develop a pattern. Challenge students to predict the perimeter of a 1 by 10 rectangle? (22) Can they tell you how to find the perimeter of a rectangle that is n squares long? ($2n + 2$)

Problems A through E might lead students to believe that as the area increases, the perimeter also increases. Problem H is a counterexample.

E, I, K, O, and P all have area 5. I has a different perimeter. Why?

D, H, J, N, and R all have area 4. H has a different perimeter. Why?

Page 2:

How can you add area without increasing perimeter?

Could you ever add area and decrease the perimeter?

What *kind* of shapes have big perimeters? Small perimeters?

Page 3:

What kinds of shapes have the smallest possible perimeter for a given area?

How can you add area with a minimal increase in perimeter? Is there a strategy you can describe?

How do you know that you've found the smallest perimeter?

Page 4:

Encourage students to use the table to look for patterns and investigate conjectures (such as those from the previous three pages.)

Page 5:

The way that the figures are labeled encourages students to notice that the perimeter is twice the sum of the sides of the rectangle.

Students see several examples of rectangles with the same area and increasing perimeters. What kinds of rectangles have the largest perimeters? The smallest?

Headline stories

...provide open-ended situations in which specific mathematical ideas are embedded.

Purposes:

- To develop students' skills at deriving real-world meaning from mathematical statements
- To develop students' skills at deriving mathematical meaning from real-world situations
- To help students learn to solve problems by understanding how they are built
- To develop students' skills at translating among natural language, pictures, and mathematical notation

Features:

- A headline story is a story—znot a problem
- A headline story usually allows for multi-level responses
- A headline story usually allows for responses in a variety of formats
- A headline story requires students to approach the situation as a whole

Examples:

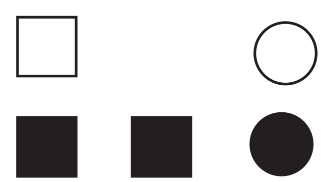
Tell a story to fit "2 + 1 = 3"

Possible responses:
- I had two stuffed dogs, and yesterday my mom gave me another one.
- It's 2:00 now, and school will be over at 3:00.
- There are three people in my family—my parents and me.

That bus is always 20 minutes late! It was supposed to arrive at ___ , but it arrived at ___ .

Possible responses:
- It was supposed to arrive at 2:30 , but it arrived at 2:50.
- If it was supposed to arrive at 11:50 at night, it would arrive after midnight.

Tell a story about this picture:



Possible responses:
- There are five shapes, three of them black and two of them white.
- There are more black shapes than white shapes.
- All the black shapes are in the second row, and all the swhite shapes are in the first row.
- If you ask me to add another shape, I'd draw a white square in the middle of the first row. Then, the first row would be the same as the second, only white.
- If you ask me to add another shape, I'd draw a white circle in the middle of the first row. Then, there would be as many circles as squares, and as many white shapes as black shapes.

Headline stories

I have ___ coins worth ___ ¢.

Possible responses:
- I have 3 coins worth 7¢, a nickel and two pennies.
- I have 2 coins worth 50¢. I must have two quarters.
- If I know how many coins and also know the total amount, can I always tell which coins I have?
- If I have 6 coins, I cannot have less than 6¢.

In my hand I have 9¢.

Possible responses:
- I must have at least 5 coins
- I could have 9 pennies
- I have 5 coins or 9 coins
- I have an odd number of coins
- If I find a penny I will have 10¢
- I can share this amount with 2 friends, if I have only pennies.
- I cannot share this amount equally with one friend.

$(6 - 1) + (6 + 1) = 12$

$(6 - 2) + (6 + 2) = 12$

$(6 - 3) + (6 + 3) = 12$

Possible responses:
- The answer is the same in all three examples.
- It is always six nimus a number plus six plus a number.
- We also can write -4, -5, and -6.
- If you add two numbers, you can make one of them smaller and the other the same amount larger, and then the result will not change.
- Six kids had no free choice time, and six other kids had it. Then the teacher allowed another kid to have a free choice time. And now only five kids did not have a free choice time. And then the teacher allowed another kid to have a free choice time...

Headline stories

Abby, Brad, and Cindy all live om Beacon Street. Abby and Brad live 3 blocks apart, and Brad and Cindy live 5 blocks apart.

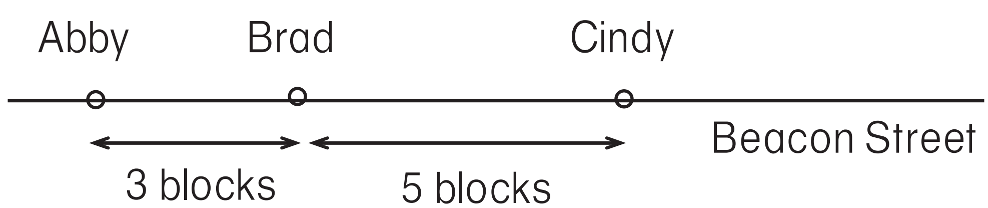
Possible responses:

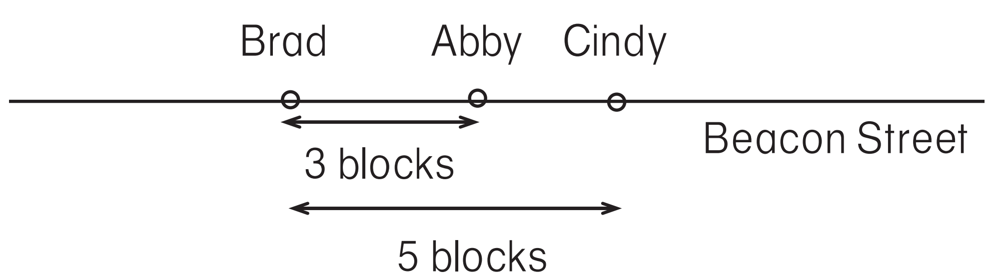
- Beacon Street is at least 5 blocks long!

- Abby and Cindy could live 8 blocks apart.

- Brad lives closer to Abby than to Cindy.

- There are two possibilities:





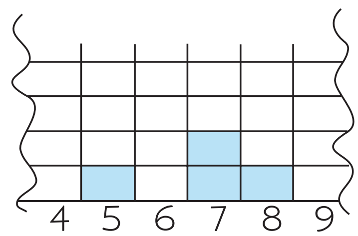
Edith's class has fewer than 40 students. Two-ninths of the students are absent, and one-fourth of the students are at music practice.

Possible responses:
- More students are at music practice than absent.
- More than half of the students are present.
- Fewer than three-quarters of the students are present.
- The number of students in Edith's class must be a multiple of 9 and a multiple of 4. So, there are 36 students in Edith's class.
- 8 students are absent, 9 students are at music practice.

Number Race

Introducing the Number Race game. Have the students work in pairs. Each pair of students needs two number cubes and a Number Race sheet. Demonstrate with one pair's materials as you explain that they will take turns rolling the cubes and finding the sum of the two numbers on top. Once they know the sum, they color in a space for that sum, starting with the bottom space. When that same sum happens on one of their next turns, they'll color the next space up.

For example, if players roll 7, 5, 8, and then 7 again, the bottom of their score sheet should look like this:



The first player rolls the cubes, computes the sum, and colors the space. While the first player is coloring, the second player rolls the cubes and computes the sum. The first number to reach the top of the chart is the winning number, and they should write that number in the winner's box.

Playing the Number Race game. As students begin to play Number Race, check quickly with all the pairs to make sure they understand the directions. Then, as you walk around to help, listen for children's predictions about what can or can't happen, or what is or isn't likely. These are statements to bring up in the class discussion. Pairs should play as many rounds of the game as they can in the time you allow, using a new Number Race sheet for each game.

Discussing the Number Race game. Gather the completed Number Race sheets and write the numbers 1–12 in a column on the board. Ask two students to help you. For each sheet, one of the helpers calls out the winning number and the other adds a check by that number on the board:

⋮	
6	✓✓
7	✓✓✓
8	✓
⋮	

When this is done, ask the class for their impressions of the game. What conclusions can they draw from the game? You might ask questions like:

- Which sums happen often (or, happen rarely)?
- Are there any numbers that can't win? (One can never win.)

Students may use their own Number Race pages along with the chart on the board to help answer the questions. Use the completed tally to see whether the results agree with any claims made in the discussion.

Extending the discussion. You may want to use Number Race as a math center or free-choice activity in your classroom. Ask students to keep all their completed game sheets in a folder, so you can collect the results of the game over many more trials. Once you have a lot of data, you may want to re-open the discussion.

Ask students to think about what happened in their games. Ask about the likelihood of some specific numbers winning. You may use the list of statements your students made during their time playing the game, or you can use some of these:

- Number 7 wins the race. (Fairly likely, probably will happen)
- Number 2 wins the race. (Rather unlikely, probably won't happen)
- The column for number 2 is three squares high before any other number. (More likely than 2 winning the race to the checkered flag, but still not likely)
- Number 1 wins the race. (Impossible, can't happen)

If students declare some number a sure winner, ask if it will win every time. Look through your data to see if you can find a case where it didn't win, or discuss why it has to win. If students declare that some event will never happen, ask why the event is so unlikely. Did it ever happen in the class's data?

You might make a table to show the class's results for the Number Race game, listing each number from 1 to 12 with the number of times it was the winner over all the trials of the game. Relate the data from your class to the probability estimates. For example: "Jill said that the likelihood that 2 would win the race was very low. In our class it never won, but in one game, 2 made it up to 6 boxes. Does our data agree with Jill's estimate?"

Students should see that 2 has occurred less often than 7. Make sure all students agree, and then ask: "Why?" If you are lucky, a student might say something like: "Two only happens when there is 1 on each of the cubes (write $2 = 1 + 1$), but seven can be made of 3 and 4, or 1 and 6." (Write $7 = 3 + 4$ and $7 = 1 + 6$.) If no one in your class suggests finding the different ways numbers can be made, use your judgment to either drop the discussion or bring it up yourself.

You might want to build on this situation, by making (with the help of your students) a chart to show all the ways to make numbers 1 to 12. Ask your class to imagine rolling a green and a blue number cube. Show all the different ways a particular sum can occur using green and blue chalk to represent the different cubes. Ask students how they know they have all the ways. (See Notes about the Math behind It for a sample chart.)

Notes about the Math behind It. In the Number Race game, students are learning and practicing addition facts up to 6 + 6, but they are also collecting and representing data, and investigating probability. This is a scientific experiment.

Your students are investigating probability informally. They're getting an idea of which outcomes are possible and which are impossible. They're also getting a feeling for which sums are more likely. It's possible to roll two cubes and get a sum of 6—for example you could roll a 4 and a 2. It's easy to roll a 7, and hard to roll a 12. It's impossible to roll two cubes and get a sum of 1, though, because the smallest number on each cube is 1, and 1 plus 1 is 2. (This kind of reasoning is an early experience with mathematical proof.)

Rather than telling your students about probability, or explaining the theory behind what they're observing, let them investigate and tell you what they discover. If you know that something they've said is false, suggest a direction for them to investigate so they can figure this out for themselves. For example, if students tell you it's impossible to roll a 2, ask them to show you the smallest number possible.

In the Number Race game, there are many possible outcomes for each roll of the number cubes. Students can explain mathematically the relative likelihoods that they have discovered experimentally. If you like, you can have them fill in an addition table—which is good practice for them anyway—that looks like this:

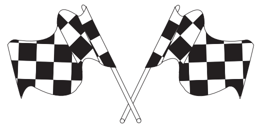
green die roll \ blue die roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There is only one 12 inside the table, because the only way to get a 12 is if both number cubes land on 6. But there are many ways to get a 7, and so 7 is often the winner! The students can't compute the probability of 10 winning, but they can begin to see that, in the table, it shows up half as often as 7, and just as often as 4, and that's probably pretty close to the results they've seen in their games.

Language and Notation. *Possible, impossible, always,* and *never* are all mathematical words, and you want your students to be careful how they use them. Point out that it may be hard to roll a 2, but it isn't impossible. Students will also use words that relate to the frequency of events, but again it's better to say, "That happens a lot," rather than "That happens all the time!" Be a little picky with your students, and increase their awareness of these ideas.

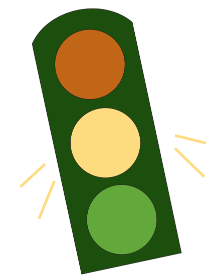
Opportunities for Ongoing Assessment. As students play the Number Race game, you can see which students have memorized many of their addition facts, which ones are counting on from the larger number, and which ones are using their fingers. You may also hear strategies like "6 is 1 more than 5, so if you have two 6s, it's like two 5s and two 1s, so it's12."

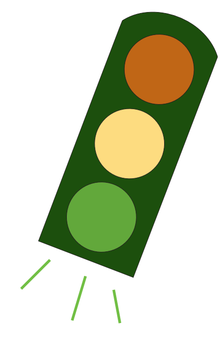
Name:______________________ Date:______________



Number Race!

WINNER!!





START!

1	2	3	4	5	6	7	8	9	10	11	12