

## Index of key problems

Since this book is a structured collection of mathematical explorations, it is less encyclopedic than the usual textbook. Though we believe it is important to work through chapters and sections in the order the problems are presented (so that the “story line” is not lost), we realize that there might be compelling reasons to review on one or more of the big picture problems. Therefore, in lieu of an index, we provide here an annotated list of the key problems presented in the text (those labeled **PROBLEM**).

*Some problem statements have been slightly modified in order to provide context.*

### CONSECUTIVE SUMS PROBLEM 3

Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can? Experiment, look for patterns, and come up with some conjectures. Write up what you find.

*It will usually be useful to also consider the activities before and after these problems, since the surrounding explorations include essential concepts and strategies for solving—or applications of—the key problems.*

### CONSECUTIVE SUM JEOPARDY 8

In your investigations of the sisters, cousins, and aunts of the Consecutive Sums Problem, what questions (if any) have you run across which have the following sets of numbers as *answers* or partial answers?

### THE HORSE PROBLEM 34

Using the horse’s tail as a unit—a square of side 1—find the area of the tangram horse shown in the figure.

### COORDINATE FORMULA FOR THE AREA OF A TRIANGLE 48

Use the (horizontal) circumscribed rectangle method to derive a formula for the computing the area of a triangle in terms of its (Cartesian) vertices.

### COMPARING THE LUNE TO THE CLAWS 55

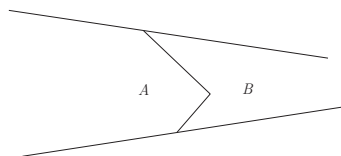
Assuming the smaller circle in the lune’s construction is congruent to the larger circle in the claws’ construction, which is bigger—the lune or the claws? Make a guess, then check by computing the two areas.

### EASY AS $\pi$ 58

Express the areas of the inscribed and circumscribed  $n$ -gons as functions of  $n$  (and  $r$ ), then use that information to approximate the area of the circle of radius  $r$  to within  $0.001r^2$ , being sure to explain your reasoning.

### DERIVING THE CIRCLE’S CIRCUMFERENCE 59

Use inscribed and circumscribed  $n$ -gons to approximate, then compute, the circumference of a circle of radius  $r$ . In the process, explain how you may conclude that the area of a circle is  $r$  times half its circumference.



For example, if  $F(7) = 35$   
and  $F(9) = 45$ , then  $F(16)$   
would be 80.

**TIMSS PROBLEM** 64

Arnold and Betty are farmers. Here is a map of part of their land.  $\triangleright$  They want to straighten out the boundary between their properties without changing the amounts of land they each have. How can they do it?

**TURN ONE INTO FOUR** 68

Show that every triangle is a 4-reptile by demonstrating how they can be cut into four congruent triangles that are similar to the original triangle.

**MYSTERY FUNCTION PROBLEM** 90

A function,  $F$ , has only real numbers in its domain (its inputs), and its range (outputs) also consists of real numbers. For any real numbers  $a$  and  $b$ ,  $F(a + b) = F(a) + F(b)$ . That is, if you put in the sum of two numbers, what comes out is the sum of the outputs for the two numbers fed in separately. What does  $F$  look like, algebraically?

**THE MEASURING CUPS PROBLEM** 106

Kathryn has a cooking pot and two measuring cups. One cup holds 4 fluid ounces, the other holds 6 fluid ounces. Neither cup has marks that allow Kathryn to measure less than these amounts. Can she measure 2 fluid ounces using these cups? Can she measure 14 fluid ounces? 7 fluid ounces? For each amount she could measure, explain how.

**RATIONAL POINTS ON THE UNIT CIRCLE** 114

There are four integer-valued points on the unit circle; namely  $(1, 0)$ ,  $(0, 1)$ ,  $(0, -1)$ , and  $(-1, 0)$ . Are there any other *rational* points on the unit circle? If so, find at least six rational points in the first quadrant that lie on the unit circle.

**THE PYTHAGOREAN THEOREM** 115

Carefully state the Pythagorean Theorem. Be sure that your statement of the theorem will be clear to *anyone*, even if they have never heard of the theorem.

**HOW ABOUT THE  $M$ -GRAPH?** 117

The  $M$ -graph for  $\overline{AB}$  is the set of all points  $M$  which are midpoints of  $\overline{AC}$  where  $\triangle ABC$  is a right triangle. What does the  $M$ -graph look like for a given  $\overline{AB}$ ? Carefully explain your conclusions and reasoning.

**EUCLID'S PROOF OF THE PYTHAGOREAN THEOREM** 123

The following paragraph (and figure) provides the gist of the argument Euclid used in *The Elements* to prove the Pythagorean Theorem (Proposition 47 from Book 1). Fill in the details.

THE LAW OF COSINES	132
Prove the conjecture (concerning the relationship between $(BC^2 + AC^2) - AB^2$ and $\cos(\angle ACB)$ ) you made in problem 3 (of Chapter 4, section 3).	
CAN PYTHAGOREAN TRIPLES EVER BE ODD?	139
How many even entries can <i>any</i> Pythagorean triple have? (Are 0, 1, 2, and 3 all possible?) Can the hypotenuse ever be the only even side length in a Pythagorean triangle?	
THE DIFFERENCE OF TWO SQUARES	146
Which counting numbers can be expressed as the difference of the squares of two counting numbers?	
PYTHAGOREAN TRIANGLES WITH INTEGER ALTITUDES	148
Describe a method to create infinitely many Pythagorean triangles having altitudes with integer length.	
SUMS AND DIFFERENCES OF SQUARES	149
Which numbers can be expressed as the <i>sum</i> of the squares of two counting numbers and also as the <i>difference</i> of the squares of two counting numbers? Describe a method for generating infinitely many such numbers.	
THE TRAIN PROBLEM	157
Given any positive integer, $n$ , derive a formula for the number of (number rod) trains of length $n$ . Explain your solution, the process you used to find it, and how you know it's correct <i>for all</i> $n$ .	
HOW MANY CARS PER TRAIN?	158
Describe a process for calculating the number of trains of length $n$ that have exactly $k$ cars.	
MS. ANTON'S PATH PROBLEM	162
Ms. Anton takes a different route (along an 8-block $\times$ 8-block grid of streets) to work every day. She will quit her job the day she has to repeat her route. If she never backtracks (she only travels north or east), how many days will she work at this job?	
THE TRAIN-PATH PROBLEM	165
Pascal's Triangle has shown up in two investigations so far: THE TRAIN PROBLEM and MS. ANTON'S PATH PROBLEM. Why? What do these two problems have to do with each other?	
PASCAL'S SUBSET THEOREM	173
Prove that if $0 < k < n$ , then ${}_n C_k = {}_{n-1} C_{k-1} + {}_{n-1} C_k$ , confirming that	

$$Pas(n, k) = {}_n C_k \text{ if } 0 \leq k \leq n.$$

$Pas(n, k)$  denotes entry  $k$  in row  $n$  of Pascal's Triangle.

## PASCAL'S FACTORIAL THEOREM 174

Use the method for proving PASCAL'S SUBSET THEOREM to prove that

$$Pas(n, k) = \frac{n!}{k!(n-k)!} \text{ if } 0 \leq k \leq n.$$

## THE BINOMIAL THEOREM 180

Complete the statement of, and then prove, the Binomial Theorem: If  $a$  and  $b$  are real (or complex) numbers and  $n$  is a non-negative integer, then

$$(a + b)^n = \sum_{k=0}^n \underline{\hspace{2cm}}$$

## HAN AND MARVIN'S PROOF: 182

In the dialogue that follows, Han and Marvin are discussing a *combinatorial* proof of the Binomial Theorem. Finish the discussion and proof for them. Be sure to provide a proof of the general result, not just the  $n = 5$  case.

## THE DICE SUM PROBLEM 190

Conjecture the value of the distribution polynomial for the possible sums when you throw  $n$  dice. Prove that your conjecture is correct.

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