# Chapter I

## *What is Mathematical Investigation?*

*Problem posing*

*Conjecture*

*Habits of mind*

*Proof*

*Problem solving*

*What if ... ?*
*What if not?*

$$1 + 2 + 3 = 6$$
$$10 + 11 + 12 + 13 + 14 + 15 + 16 = 91$$
$$3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 75$$
$$8 + 9 + 10 + 11 + 12 + \cdots + 107 = 5750$$
$$125 + 126 + 127 + 128 + \cdots + 2003 = 1999256$$
$$n + (n + 1) + (n + 2) + \cdots + (n + k) = 137$$

## Introduction

Current mathematics curricula ask students, from time to time, to "investigate." But how does one do that?

Another question: why all the fuss about investigation? Investigation is not the only way to learn mathematics, nor even the best way in *every* situation.

The ability to investigate a situation is, in itself, an important skill for students to acquire. In mathematics—as in science, or diagnosing the ills of an automobile, a computer, or a person—proper investigation is often the first step in successful problem solving. Furthermore, investigation helps to bring to the fore an essential feature of the subject itself.

Mathematics is a specialization of many of the most powerful thinking techniques people normally use. Part of its great power derives from the facts, formulas, and techniques it provides to the sciences. What makes it of value even to those who will someday forget the facts and formulas is that it highlights, extends, and refines the kinds of thinking that people do in *all* fields. These include investigation, pattern-seeking, and proof.

*Proof outside of mathematics is different, in ways, from proof within the discipline, but the fact that the same word is used attests to the relatedness of the many purposes of proof, and even to similarities in the ways of thinking.*

Skilled investigators in any field have strategies that go beyond poking around and hoping for the best. In investigation, as in other aspects of thinking, mathematics adds its own special features. What makes an investigation *mathematical*? What's next after finding a great pattern?

In *What is Mathematical Investigation?*, you will take a mathematical investigation from start to finish—from exploratory stages through reporting logically connected results—and you will find strategies that you can use with your students to develop their investigative skills.

You might also encounter some new mathematical facts and relationships, but the real purpose is for you to investigate *investigation*.

## 1. Problem solving and problem posing

Take 10 to 15 minutes for a preliminary exploration of the problem below—just long enough to develop some initial conjectures.

Such a tiny amount of time is not nearly adequate for a thorough look at this investigatory problem, but even 15 minutes should give you a sense of what students begin to see as they explore it. For the moment, this glimpse is enough.

*Students in the first year course of the* Interactive Mathematics Program (IMP) *are given three days to explore this lovely problem. You will get the chance to explore this problem in greater depth later.*

In the brief time you devote to the problem now, keep track of partial answers and any new questions that may come up.

**PROBLEM**

**1.** The number 13 can be expressed as a sum of two consecutive counting numbers, $6+7$. Fourteen can be expressed as $2+3+4+5$, also a sum of consecutive counting numbers.

*Positive integers go by many aliases: the counting numbers, the natural numbers, $\mathbb{Z}^+$.*

THE CONSECUTIVE SUMS PROBLEM:
Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can?

Experiment, look for patterns, and come up with some conjectures. Write up what you find.

**Remember: For now, take only 10 to 15 minutes.**



### Dissecting the problem

To investigate a problem well, you should get right to its heart. The first two sentences of problem 1 just say what is meant by "sum of consecutive counting numbers," and the last two sentences are merely guidance for the student. The problem's essence is in the middle two sentences:

> Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can?

Even this can be boiled down. The real information is:

> ... counting numbers ... expressed as sum of two or more consecutive counting numbers ...

**2.** Concealed within that deceptively simple boiled-down version are at least five essential features of the problem. Two are given to you. Find *at least three* others.

(a) It is about a *sum*.

(b)

(c)

(d)

(e) There are restrictions that the problem *could* make, but *does not*. The fact that it *fails* to make more restrictions is part of what makes it *this* problem and not another.

*This, of course, is a feature of every problem. Learning to notice what is* not *stated is extremely hard for everybody.*

Feature (e) may seem almost too silly to list, but it is important! For example, the problem refers to a "sum of two or more consecutive counting numbers." A more restrictive problem might ask "Which numbers can (or cannot) be expressed as a sum of exactly three consecutive counting numbers?"

*This is a* new *problem, and an interesting one!*

Similarly, the problem asks which numbers can be expressed at all, in any number of ways. A more specific problem might ask "Which numbers can (or cannot) be expressed in exactly one way (or two or . . . ) as a sum of consecutive counting numbers?"

*This is* another *new and interesting problem!*

### Modifying the problem

**3.** By yourself or with others, brainstorm to see what related problems evolve from this one as you change the features one (or at most two) at a time. Write down and share this set of new problems.

*Two great problems are listed in the previous paragraphs, but they are not the only good ones that come from changing features of the original problem.*
*How do you decide, before investigating, which will be a worthwhile problem to pursue? Is it intuition? Experience? What goes into your decision?*

**4.** Pick one or more of your problems and explore them just long enough to build some preliminary conjectures.

**For now, take only 10 to 15 minutes.** As before, you won't have enough time for a real investigation, but you should get a rough idea of what the problem has in store.

**Problem-posing strategies**

Problem 3 asked you to "change the features," but *how* should that be done? Are there any reliable ways to do that and get "good" problems as a result?

As you gain experience, you'll develop your own set of tricks for modifying the features of a problem, but here are four that are almost always among the most useful.

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**i. Make a feature more restrictive:** If the problem is about triangles, restrict it to right (or scalene or ...) triangles. If the problem uses a calculation that involves two or more numbers, restrict it to *exactly* two (or three or ...).

*This is sometimes referred to as finding* special cases.

**ii. Relax a feature:** If the problem is about right triangles, see how it changes if you allow *all* kinds of triangles, or maybe all polygons. If the problem uses a restricted subset of numbers (e.g., only {1, 2, 3, ...}), see what happens when you expand that set in various ways.

*This is sometimes referred to as* generalizing*, or extending the domain.*

**iii. Alter the details of a feature:** If the problem concerns right triangles, see how it changes if you choose acute triangles. If the problem calls for one set of numbers (e.g., {1, 2, 3, ...}), try a different set (e.g., {1, 3, 5, 7, ...} or {0, 3, 6, 9, ...} or {0.5, 1, 1.5, 2, ...}). If the problem uses arithmetic operations, see what happens if you systematically alter them (e.g., substituting $+$ and $-$ for $\times$ and $\div$ or vice versa), and if it specifies equality, see what happens if you require a specific inequality (e.g., $>$).

*These modifications may* change the domain *of a problem, or alter a parameter.*

**iv. Check for uniqueness:** If the problem only asks *if* something can be done, ask if (or when) it can be done in *only one way*.

*Asking* how many ways can this be done? *is often productive.*

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**5.** Apply these and your own rules to generate interesting variants on the following problem:

"How many triangles with perimeter 12 and integer side lengths can you construct?"

*You get to be the judge of what is an interesting variant of the problem.*

**6.** Now, go back to the CONSECUTIVE SUMS PROBLEM. Look over the list of features you made for problem 2 and see if applying these rules to each of the features gives you any new problems.

## Ways to think about it

The statement of each problem (or a paraphrased version) is provided in the margin for your convenience.

THE CONSECUTIVE SUMS PROBLEM:
*Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can? Make a conjecture.*

1. Look at a lot of examples, being sure to keep track of everything — maybe make a table. Once you get a tentative conjecture, check it out with a few more examples. Remember that you're not being asked to prove anything. Carefully write down and share your observations and predictions.

***Problem:*** *There are at least five essential features of the consecutive sums problem. One is "It is about a sum." Find* at least three *others.*

2. Analyze the problem statement and context. What "kind" of sum are we talking about here (what's being added)? Is the type or number of addends restricted? How about the relationship between addends?

***Problem:*** *Brainstorm to see what related problems evolve from this one as you change the features one (or at most two) at a time.*

3. First, think of some ways you might alter the original problem to create a new one. One possibility is by adding or removing restrictions. Can you think of other alterations?

***Problem:*** *Pick one or more of the problems you created in problem 3 and explore them just long enough to build some preliminary conjectures.*

4. As in problem 1, we're looking for conjectures, not proofs. Try some experimentation and see what you can come up with.

***Problem:*** *Apply these and your own rules to generate interesting variants on the following problem:*
*"How many triangles with perimeter 12 and integer side lengths can you construct?"*

5. Go through each of the suggested ways of altering problems and see how you might modify this triangle problem.

***Problem:*** *Now, go back to the consecutive sums problem. Look over the list of features you made for problem 2 and see if applying these rules to each of the features gives you any new problems.*

6. Did you miss anything when you modified the consecutive sum problem in problem 3?