Patterns in Pascal's Triangle

Project Statement

(1) How many odd numbers are in the 100th row of Pascals triangle?
(2) How many entries in the 100th row of Pascals triangle are divisible by 3? By 5?

When you divide a number by 3, the remainder is either 0, 1, or 2. Divide the entries in Pascals triangle by 3 and color them accordingly. You get a beautiful visual pattern. Can you explain it? Can you generate the pattern on a computer?

What about the patterns you get when you divide by other numbers?

Prerequisites

Warm Up Problems

Note!
We do not include results for these warm up problems on the Making Mathematics Web site. If you need assistance, please write to us at our project mailbox.

Hints

the 12242448 pattern
the geometric approach

Resources

Teaching Notes

Introducing the Project.

Working on the Project.

Closure.

Results

There are at least two approaches to this project: an arithmetic and a geometric approach. Each approach informs the other, so, while we present them separately, you'll see many interdependencies.

An Arithmetic Approach. There are eight odd numbers in the 100th row of Pascal's triangle, 89 numbers that are divisible by 3, and 96 numbers that are divisible by 5.

Of course, one way to get these answers is to write out the 100th row, of Pascal's triangle, divide by 2, 3, or 5, and count (this is the basic idea behind the geometric approach).

But let's see if we can find a more efficient (and elegant) way to get our answers. In an attempt to show how one might actually come up with interesting results, what follows is a collage of various arithmetic approaches that we've seen over the years in our work with students and teachers.

We want to count the number of elements in a row of Pascal's triangle that are (or are not) divisible by some integer. Divisibility by a prime is easier to deal with, so let's look at that case. The question, then, is

> Given a non-negative integer n and a prime p, which numbers $\binom{n}{k}$ are divisible by p?

Well, sometimes a more pointed question is easier to answer:

> Given a non-negative integer n and a prime p, what is the highest power of p that divides $\binom{n}{k}$?

This "highest power of p" function is useful in many contexts, and it's sometimes denoted by ord_p, so, for example:

$$\begin{aligned} \text{ord}_5(40) &= 1 \qquad (20 = 5^1 \cdot 2^3) \\ \text{ord}_2(40) &= 3 \\ \text{ord}_7(40) &= 0 \\ \text{ord}_7(5^8 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11^5) &= 2 \end{aligned}$$

To get $\text{ord}_p(n)$, factor n into primes and look at the power of p that shows up. That's $\text{ord}_p(n)$. You can check that ord_p has the property that, for non-negative integers m and n,

$$\text{ord}_p(mn) = \text{ord}_p(m) + \text{ord}_p(n)$$

And, if n is a factor of m,

$$\text{ord}_p\left(\frac{m}{n}\right) = \text{ord}_p(m) - \text{ord}_p(n)$$

So, our question becomes

> Given a non-negative integer n and a prime p, what's an easy way to calculate $\text{ord}_p\left(\binom{n}{k}\right)$?

In particular, we want to know when

$$\operatorname{ord}_p\left(\binom{n}{k}\right) > 0$$

because this is when p is a factor of $\binom{n}{k}$. Of course, what we really want to do is to count all the entries in the nth row of Pascal that have $\operatorname{ord}_p > 0$, but let's worry about that later.

Well, there's an explicit formula for $\binom{n}{k}$ in terms of factorials:

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

So, using the properties of ord (the ord of a quotient is the difference of the ords, and the ord of a product is the sum of the ords), we have

$$\begin{aligned}\operatorname{ord}_p\left(\binom{n}{k}\right) &= \operatorname{ord}_p\left(\frac{n!}{k! \cdot (n-k)!}\right) \\ &= \operatorname{ord}_p(n!) - \operatorname{ord}_p(k!) - \operatorname{ord}_p\left((n-k)!\right)\end{aligned}$$

It seems like a next step might be to find a way to find the ord of a factorial. A numerical example points the way: let's calculate $\operatorname{ord}_3(139!)$.

So, we want to find the highest power of 3 that divides

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 137 \cdot 138 \cdot 139$$

Each multiple of 3 give us at least one "contribution":

$$3, 6, 9, 12, 15, \ldots$$

This stops at the last multiple of 3 before 139 (that is, at 138), and there are about $\frac{139}{3}$ numbers in the list. In fact, there are *exactly*

$$\left\lfloor \frac{139}{3} \right\rfloor = 46$$

where $\left\lfloor \frac{139}{3} \right\rfloor$ means the "integer part" of $46\frac{1}{3}$ (that is, the integer part of $\frac{139}{3}$).

Of course, we've missed some "extra" multiples of 3: every multiple of 9 (that is, $9, 18, 27 \ldots$) counts *twice*, and there are

$$\left\lfloor \frac{139}{9} \right\rfloor = 15$$

of these. But, we've now missed the extra multiples of 3 that come from multiples of 27, and there are

$$\left\lfloor \frac{139}{27} \right\rfloor = 5$$

of these. You get the idea. Next, we need to count the multiples of 81 (to get the extra factor that comes from 3^4), and there are

$$\left\lfloor \frac{139}{81} \right\rfloor = 1$$

of these (namely, 81 itself). If we didn't know any better, we could count the multiples of 243 that are less than 139, and there'd be

$$\left\lfloor \frac{139}{243} \right\rfloor = 0$$

of these. And everything would be 0 from here on. So, the power of 3 that divides 139! is

$$\text{ord}_3(139!) = 46 + 15 + 5 + 1 = 67$$

remember where this came from. Summarizing our calculations, we could say:

$$\text{ord}_3(139!) = \left\lfloor \frac{139}{3} \right\rfloor + \left\lfloor \frac{139}{3^2} \right\rfloor + \left\lfloor \frac{139}{3^3} \right\rfloor + \left\lfloor \frac{139}{3^4} \right\rfloor + \left\lfloor \frac{139}{3^5} \right\rfloor + \left\lfloor \frac{139}{3^6} \right\rfloor + \left\lfloor \frac{139}{3^7} \right\rfloor + \dots$$

Of course, after awhile (in fact, after 3^4), all these "floors" are 0. So, we have a method for finding $\text{ord}_p(n!)$:

Proposition 1. *If n is a non-negative integer and p is a prime, then*

$$ord_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \left\lfloor \frac{n}{p^4} \right\rfloor + \left\lfloor \frac{n}{p^5} \right\rfloor + \left\lfloor \frac{n}{p^6} \right\rfloor + \left\lfloor \frac{n}{p^7} \right\rfloor + \dots$$

This unsatisfying in many ways. For one, it doesn't tell you when the terms of the sum drop off to zero. For that, you'd have to write n in base p. But wait—if you write n in base p, the whole sum in the proposition gets easier. Here's how:

Let's go back to our example if 139 and 3. First, let's write 139 as a base-3 expansion:

$$139 = (1 \cdot 1) + (1 \cdot 3) + (0 \cdot 3^2) + (2 \cdot 3^3) + (1 \cdot 3^4)$$

So,

$$139_3 = 12011$$

Then, to get $\text{ord}_3(139!)$, using we use the proposition and calculate as follows:

$$\left\lfloor \frac{139}{3} \right\rfloor = \left\lfloor (1 \cdot \frac{1}{3}) + (1 \cdot 1) + (0 \cdot 3) + (2 \cdot 3^2) + (1 \cdot 3^3) \right\rfloor = (1 \cdot 1) + (0 \cdot 3) + (2 \cdot 3^2) + (1 \cdot 3^3)$$

$$\left\lfloor \frac{139}{3^2} \right\rfloor = \left\lfloor (1 \cdot \frac{1}{3^2}) + (1 \cdot \frac{1}{3}) + (0 \cdot 1) + (2 \cdot 3) + (1 \cdot 3^2) \right\rfloor = (0 \cdot 1) + (2 \cdot 3) + (1 \cdot 3^2)$$

$$\left\lfloor \frac{139}{3^3} \right\rfloor = \left\lfloor (1 \cdot \frac{1}{3^3}) + (1 \cdot \frac{1}{3^2}) + (0 \cdot \frac{1}{3}) + (2 \cdot 1) + (1 \cdot 3) \right\rfloor = (2 \cdot 1) + (1 \cdot 3)$$

$$\left\lfloor \frac{139}{3^4} \right\rfloor = \left\lfloor (1 \cdot \frac{1}{3^4}) + (1 \cdot \frac{1}{3^3}) + (0 \cdot \frac{1}{3^2}) + (2 \cdot \frac{1}{3}) + (1 \cdot 1) \right\rfloor = (1 \cdot 1)$$

And everything is 0 after this. Wow, look at this. Look at the rightmost equations. One thing we could do is to read down instead of across; that would allow us to add "like" powers of 3. And, just like when you read across, when you read down you see the base-3 digits of 139, first all except the "units" 1, then all except the 11, then all but the 011, and finally all but the 2011. Will this always happen? Let's see.

Suppose

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_s p^s$$

Then

$$\begin{aligned}\left\lfloor \frac{n}{p} \right\rfloor &= \left\lfloor (n_0 \cdot \frac{1}{p}) + (n_1 \cdot 1) + (n_2 \cdot p) + (n_3 \cdot p^2) + \cdots + (n_s \cdot p^{s-1}) \right\rfloor \\ &= (n_1 \cdot 1) + (n_2 \cdot p) + (n_3 \cdot p^2) + \cdots + (n_s \cdot p^{s-1})\end{aligned}$$

and

$$\begin{aligned}\left\lfloor \frac{n}{p^2} \right\rfloor &= \left\lfloor (n_0 \cdot \frac{1}{p^2}) + (n_1 \cdot \frac{1}{p}) + (n_2 \cdot 1) + (n_3 \cdot p) + \cdots + (n_s \cdot p^{s-2}) \right\rfloor \\ &= (n_2 \cdot 1) + (n_3 \cdot p) + \cdots + (n_s \cdot p^{s-2})\end{aligned}$$

And, more generally,

$$\begin{aligned}\left\lfloor \frac{n}{p^k} \right\rfloor &= \left\lfloor (n_0 \cdot \frac{1}{p^k}) + (n_1 \cdot \frac{1}{p^{k-1}}) + (n_2 \cdot \frac{1}{p^{k-2}}) + (n_3 \cdot \frac{1}{p^{k-3}}) + \cdots + (n_s \cdot p^{s-k}) \right\rfloor \\ &= (n_k \cdot 1) + (n_{k+1} \cdot p) + \cdots + (n_s \cdot p^{s-k})\end{aligned}$$

But

$$\text{ord}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \left\lfloor \frac{n}{p^4} \right\rfloor + \left\lfloor \frac{n}{p^5} \right\rfloor + \left\lfloor \frac{n}{p^6} \right\rfloor + \left\lfloor \frac{n}{p^7} \right\rfloor + \ldots$$

So let's write this all out and see if we can use the "add down" trick:

$$
\begin{aligned}
\left\lfloor \frac{n}{p} \right\rfloor &= (n_1 \cdot 1) + (n_2 \cdot p) + (n_3 \cdot p^2) + (n_4 \cdot p^3) + \cdots + (n_s \cdot p^{s-1}) \\
\left\lfloor \frac{n}{p^2} \right\rfloor &= (n_2 \cdot 1) + (n_3 \cdot p) + (n_4 \cdot p^2) + (n_5 \cdot p^3) \cdots + (n_s \cdot p^{s-2}) \\
\left\lfloor \frac{n}{p^3} \right\rfloor &= (n_3 \cdot 1) + (n_4 \cdot p) + (n_5 \cdot p^2) + \cdots + (n_s \cdot p^{s-3}) \\
\vdots \quad & \quad \quad \vdots \\
\left\lfloor \frac{n}{p^s} \right\rfloor &= (n_s \cdot 1)
\end{aligned}
$$

If we add down, we get the "clipped" sum of the base-p digits of n. Let's invent a notation for this: Let $\sigma_p(n)$ be the sum of the base-p digits of n. Then adding down, we have

$$
\begin{aligned}
\mathrm{ord}_p(n!) = (\sigma_p(n) - n_0) \cdot 1 + (\sigma_p(n) - n_0 - n_1) \cdot p + (\sigma_p(n) - n_0 - n_1 - n_2) \cdot p^2 + \\
(\sigma_p(n) - n_0 - n_1 - n_2 - n_3) \cdot p^3 + \cdots + (\sigma_p(n) - n_0 - n_1 - n_2 - \cdots - n_{s-1}) \cdot p^{s-1} + \\
(\sigma_p(n) - n_0 - n_1 - n_2 - \cdots - n_s) \cdot p^s
\end{aligned}
$$

The last line is 0, but we put it in to keep the pattern going. Time for some algebraic fooling around. Gather the $\sigma_p(n)$s, the n_0s, the n_1s, and so on:

$$
\begin{aligned}
\mathrm{ord}_p(n!) = \sigma_p(n) &\left(1 + p + p^2 + \cdots + p^{s-1}\right) \\
&- n_0 \left(1 + p + p^2 + \cdots + p^{s-1}\right) \\
&- n_1 \left(p + p^2 + \cdots + p^{s-1}\right) \\
&- n_2 \left(p^2 + \cdots + p^{s-1}\right) \\
&\quad \vdots \\
&- n_s \left(p^s\right)
\end{aligned}
$$

Each sum is a geometric series, just begging to be summed:

$$\begin{aligned} \mathrm{ord}_p(n!) = \sigma_p(n) \left(\frac{p^s - 1}{p-1}\right) \\ - n_0 \left(\frac{p^s - 1}{p-1}\right) \\ - n_1 \left(\frac{p^s - p}{p-1}\right) \\ - n_2 \left(\frac{p^s - p^2}{p-1}\right) \\ \vdots \\ - n_s \left(\frac{p^s - p^s}{p-1}\right) \end{aligned}$$

Factor out the denominator of $p-1$ and rearrange what's left:

$$\begin{aligned} \mathrm{ord}_p(n!) = \frac{1}{p-1} \big(p^s \sigma_p(n) - p^s(n_0 + n_1 + \cdots + n_s) \\ -\sigma_p(n) + n_0 + n_1 p + n_2 p^2 + \cdots + n_s p^s\big) \end{aligned}$$

Oh, how nice: since

$$\begin{aligned} n_0 + n_1 + \cdots + n_s = \sigma_p(n) \qquad \text{and} \\ n_0 + n_1 p + n_2 p^2 + \cdots + n_s p^s = n \end{aligned}$$

this simplifies to

$$\frac{n - \sigma_p(n)}{p-1}$$

Let's state it as a result:

Proposition 2. *If n is a non-negative integer and p is a prime,*

$$\mathit{ord}_p(n!) = \frac{n - \sigma_p(n)}{p-1}$$

where $\sigma_p(n)$ is the sum of the digits in the base-p expansion of n.

We're in a better position now to evaluate $\mathrm{ord}_p\left(\binom{n}{k}\right)$. In fact,

$$\begin{aligned}\mathrm{ord}_p\left(\binom{n}{k}\right) &= \mathrm{ord}_p\left(\frac{n!}{k!\cdot(n-k)!}\right)\\ &= \mathrm{ord}_p(n!) - \mathrm{ord}_p(k!) - \mathrm{ord}_p\left((n-k)!\right)\\ &= \frac{n-\sigma_p(n) - (k-\sigma_p(k)) - ((n-k)-\sigma_p(n-k))}{p-1}\\ &= \frac{\sigma_p(k)+\sigma_p(n-k)-\sigma_p(n)}{p-1}\end{aligned}$$

This is pretty interesting. Let's state it as a result:

Proposition 3. *If n is a non-negative integer and p is a prime,*

$$ord_p\left(\binom{n}{k}\right) = \frac{\sigma_p(k)+\sigma_p(n-k)-\sigma_p(n)}{p-1}$$

where $\sigma_p(n)$ is the sum of the digits in the base-p expansion of n.

Notice that what we really care about is which $\binom{n}{k}$ have non-zero ords. But it turns out that one can get the exact value of the expression in proposition 3 with very little extra work (we know this because we worked it out before we wrote up these results). The fact that

$$\mathrm{ord}_p\left(\binom{n}{k}\right) = \frac{\sigma_p(k)+\sigma_p(n-k)-\sigma_p(n)}{p-1}$$

is one of those strange results that contains a surprise: We know that $\mathrm{ord}_p\left(\binom{n}{k}\right)$ is a non-negative integer, so $p-1$ must be a factor of $\sigma_p(k)+\sigma_p(n-k)-\sigma_p(n)$—that's not at all obvious. And shouldn't the digit sum of n *equal* the digit sum of k plus the digit sum of $n-k$? After all, k and $n-k$ add up to n, so shouldn't the sum of the digits work the same way? Let's see what happens with an example:

Let's look at $\mathrm{ord}_3\left(\binom{139}{32}\right)$:

Well, according to the proposition, we need to look at the digit sum (base 3) of 139, 32, and $139 - 32 = 107$. We have

$$\begin{aligned} 139 &= (1 \cdot 1) + (1 \cdot 3) + (0 \cdot 3^2) + (2 \cdot 3^3) + (1 \cdot 3^4) \\ 32 &= (2 \cdot 1) + (1 \cdot 3) + (0 \cdot 3^2) + (1 \cdot 3^3) + (0 \cdot 3^4) \\ 107 &= (2 \cdot 1) + (2 \cdot 3) + (2 \cdot 3^2) + (0 \cdot 3^3) + (1 \cdot 3^4) \end{aligned}$$

So,

$$\begin{aligned} 139_3 &= 12011 \\ 32_3 &= 01012 \\ 107_3 &= 10222 \end{aligned}$$

So,

$$\begin{aligned} \sigma_3(139) &= 5 \\ \sigma_3(32) &= 4 \\ \sigma_3(107) &= 7 \end{aligned}$$

and proposition 3 says that

$$\mathrm{ord}_3\left(\binom{139}{32}\right) = \frac{7+4-5}{3-1} = \frac{6}{2} = 3$$

And sure enough,

$$\begin{aligned} \binom{139}{32} &= 2979445870004425014061856773566 0 \\ &= 2^2 \cdot 3^3 \cdot 5^1 \cdot 11^2 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 37^1 \cdot 41^1 \cdot 43^1 \cdot 59^1 \cdot 61^1 \cdot 67^1 \cdot 109^1 \cdot \\ &\qquad 113^1 \cdot 127^1 \cdot 131^1 \cdot 137^1 \cdot 139^1 \end{aligned}$$

So, why is $\sigma_3(32)+\sigma_3(107)-\sigma_3(139)$ not 0? If we look at the traditional way we add base-3 numbers (or numbers in any base), we see where the discrepancy shows up. Here's how you'd add 32 to 107 to get 139 in base 3:

$$\begin{array}{cccccc} & & {}^1 & {}^1 & {}^1 & \\ & 0 & 1 & 0 & 1 & 2 \\ + & 1 & 0 & 2 & 2 & 2 \\ \hline & 1 & 2 & 0 & 1 & 1 \end{array}$$

So, what disturbs the digit sums are the "carries;" *that's* why

$$\sigma_3(32) + \sigma_3(107) \neq \sigma_3(139)$$

Notice that each carry gets multiplied by 3 when it jumps into the next column. For example, looking at the rightmost column, the $2+2$ is 11, so the carried 1 is really $1 \cdot 3$ (or 10 in base 3). Let's look at the general situation:

Suppose we are looking at $\binom{n}{k}$. We write n, k, and $n-k$ in base p:

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_s p^s$$
$$k = k_0 + k_1 p + k_2 p^2 + \cdots + k_s p^s$$
$$n - k = m_0 + m_1 p + m_2 p^2 + \cdots + m_s p^s$$

And suppose the carries are denoted by δ_i:

$$\begin{array}{cccccc} & {\scriptstyle \delta_{s-1}} & {\scriptstyle \dots} & {\scriptstyle \delta_1} & {\scriptstyle \delta_0} & \\ & m_s & \dots & m_2 & m_1 & m_0 \\ + & k_s & \dots & k_2 & n_1 & k_0 \\ \hline & n_s & \dots & n_2 & n_1 & n_0 \end{array}$$

Well, by the way this algorithm works,

$$\begin{aligned} m_0 + k_0 &= n_0 + p\delta_0 \\ m_1 + k_1 + \delta_0 &= n_1 + p\delta_1 \\ m_2 + k_2 + \delta_1 &= n_2 + p\delta_2 \\ &\vdots \qquad \vdots \\ m_s + k_s + \delta_{s-1} &= n_s \end{aligned}$$

Then

$$\begin{aligned} m_0 + k_0 - n_0 &= p\delta_0 \\ m_1 + k_1 - n_1 &= p\delta_1 - \delta_0 \\ m_2 + k_2 - \delta_0 &= p\delta_2 - \delta_1 \\ &\vdots \qquad \vdots \\ m_s + k_s - n_s &= -\delta_{s-1} \end{aligned}$$

Add down:

$$(m_0 + m_1 + m_2 + \cdots + m_s) + (k_0 + m_1 + k_2 + \cdots + k_s) - (n_0 + n_1 + n_2 + \cdots + n_s) = (p-1)(\delta_0 + \delta_1 + \delta_2 + \cdots + \delta_s)$$

or

$$\sigma_p(n-k) + \sigma_p(k) - \sigma_p(n) = (p-1)(\delta_0 + \delta_1 + \delta_2 + \cdots + \delta_s)$$

There's the factor of $p-1$. Dividing by it, and combining the result with the result of proposition 3, we have "Kummer's carry theorem:"

Theorem 1. *If n is a non-negative integer and p is a prime,*

$$ord_p\left(\binom{n}{k}\right)$$

is the number of carries you get when k and $n-k$ are added in base p.

Let's go back to the original question:

> Given a non-negative integer n and a prime p, for how many k is $\mathrm{ord}_p\left(\binom{n}{k}\right) > 0$?

In light of theorem 1 this is the same question as:

> Given a non-negative integer n and a prime p, for how many k is there a carry when k and $n-k$ are added in base p?

Instead of answering this, we found it easier to figure out when there's *not* a carry. And from here, we can subtract (from $n+1$) to get the desired answer.

An example will help. Suppose $p = 5$ and, $n = 133866$, so, in base 5, n is

$$13240431$$

How can two numbers add to this with *no* carries? Reading from right to left, there are two choices for the first digit—0 and 1. If k "ends" in 0, $n-k$ ends in 1 and *vice-versa*. For the "fives" place, there are 4 choices for k—it could have a 0, 1, 2, or 3 in the 5's place (and $n-k$ would have, respectively, a 3, 2, 1, or 0 in the same place). And so it goes. Here's a summary of what can happen:

place	digit in n	number of choices for digits for k
1	1	2—namely, {0,1}
5	3	4—namely, {0,1,2,3}
5^2	4	5—namely, {0,1,2,3,4}
5^3	0	1—namely, {0}
5^4	4	5— namely, {0,1,2,3,4}
5^5	2	3—namely, {0,1,2}
5^6	3	4—namely, {0,1,2,3 }
5^7	1	2—namely, {0,1}

In general, we have the answer to our question:

Theorem 2. *The number of entries in the nth row of Pascal's triangle that are* not *divisible by a prime p can be determined as follows:*

- *Write n in base p:*

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_s p^s$$

- *The number in question is*

$$\prod_{k=0}^{s} (1 + n_k) = (1 + n_0)(1 + n_1)(1 + n_2) \ldots (1 + n_s)$$

So, for example, suppose $p = 2$. Then, looking at 100 in base 2:

$$100 = (1 \cdot 2^6) + (1 \cdot 2^5) + (0 \cdot 2^4) + (0 \cdot 2^3) + (1 \cdot 2^2) + (0 \cdot 2^1) + (0 \cdot 2^0)$$

so

$$100_2 = 1100100$$

Add 1 to each digit and multiply the answers together:

$$(1 + 1)(1 + 1)(0 + 1)(0 + 1)(1 + 1)(0 + 1)(0 + 1) = 8$$

and 8 numbers in the 100th row are not divisible by 2 (that is, are odd). Try it for 3:

$$100 = (1 \cdot 3^4) + (0 \cdot 3^3) + (2 \cdot 3^2) + (0 \cdot 3^1) + (1 \cdot 3^0)$$

so

$$100_3 = 10201$$

and the number of entries in the 100th row that are *not* divisible by 3 is

$$(1 + 1)(0 + 1)(2 + 1)(0 + 1)(1 + 1) = 12$$

But there are 101 entries in the 100th row, so
$101 - 12 = 89$ are divisible by 3.

Theorem 1 is due to Ernst Kummer (1810–1893), a mathematician who laid the groundwork for some of the mathematics that led to the proof of Fermat's Last theorem. Indeed, knowing the power of a prime that divides any particular entry of Pascal's triangle turns out to be an essential tool in that proof, and more generally, in all of number theory, so this project connects to some frontline mathematics.

A Geometric Approach. We'll look at the pictures, like on
`http://www.cs.washington.edu/homes/jbaer/classes/blaise/bigblaise.html`
And, i have a paper written by kids from 15 years ago.

Extensions

The p-adic approach developed (without these words) by last summer's kids.